

Stochastic Perron for Stochastic Target Problems

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Abstract In this paper, we adapt stochastic Perron’s method to analyze stochastic target problems in a jump diffusion setup, where the controls are unbounded. Since classical control problems can be analyzed under the framework of stochastic target problems (with unbounded controls), we use our results to generalize the results of Bayraktar and Sîrbu (SIAM Journal on Control and Optimization, 2013) to problems with controlled jumps.

Keywords Stochastic target problems · Stochastic Perron’s method · Jump diffusion processes · Viscosity solutions · Unbounded controls

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1 Introduction

Introduced by the seminal papers [1], [2] and [3], the stochastic target problem is a new type of optimal control problem. The aim is to drive a controlled diffusion to a given target at a pre-specified terminal time by choosing an appropriate admissible control. The above papers and their generalizations [4,5] (to jump diffusions), [6] (to unbounded controls) provide a characterization of the associated value function as

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a viscosity solution to a non-linear Hamilton-Jacobi-Bellman (HJB) equation using the geometric dynamic programming principle proved in [2].

In this paper, our goal is to provide an analysis of this problem using stochastic Perron's method. This method was introduced in [7, 8, 9] for classical control problems. This method is a verification approach (without requiring smoothness) in that it does not use the dynamic programming principle to show that the value function is a viscosity solution. The idea is to build two classes of functions that envelope the value function and that are stable enough under minimization and maximization, respectively. This construction helps us demonstrate that the supremum over the first class is a lower semi-continuous viscosity super-solution and the infimum over the second class (the functions larger than the value function) is an upper semi-continuous viscosity sub-solution. Assuming that a comparison principle holds, we show that the infimum over the second class and the supremum over the first class (which sandwich the value function) are equal, and hence, the value function is the unique viscosity solution. Since we only work with the envelopes, not the value function itself, we never use the dynamic programming principle (and hence the measurable selection theorem). In fact, the dynamic programming principle is a corollary of our result. As pointed out by [10] and the references therein, the rigorous proof of the dynamic programming principle for controlled diffusion processes is difficult and contains subtle technical issues. Our result can be seen as an elementary alternative based only on Itô's Lemma and the comparison principle, which also has to be proved to identify the value function as the unique viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation (PDE).

We choose to work with the most general stochastic target setup from [5]. Our controls are unbounded and the controlled processes are jump diffusions. The main reason for using unbounded controls is that it allows us to use the embedding result of [11], which converts an ordinary control problem into a stochastic target problem with unbounded admissible controls. Using this result, we generalize [9] to the setting of controlled jumps.

In contrast to [9], we analyze stochastic target problems in this paper. The main contribution is the construction of the sets of stochastic semi-solutions, which are appropriate for stochastic target problems. This makes the proofs of the viscosity properties of the value function different. We also generalize our earlier result in [12] in the sense that we consider unbounded controls and controlled jumps. The presence

of the jumps and the unbounded control set brings new technical difficulties: in contrast to [12], the relaxed semi-limits are introduced for the PDE characterization, which have a nontrivial impact on the formulation of the associated PDEs and the derivation of viscosity properties of the value function using stochastic Perron's method, especially at the boundary. Of particular importance is the relaxation with respect to the test function, which appears because we consider jumps.

The rest of the paper is organized as follows. The setup of the problem, the related HJB equation and the definitions of the stochastic semi-solutions are introduced in Section 2. In Sections 3 and 4, we prove the viscosity properties in the parabolic interior and at the boundary, respectively. In Section 5, we use the comparison principle to close the gap between the viscosity super-solution and sub-solution and demonstrate the uniqueness of the viscosity solution to the associated HJB equation. In Section 6, we see how an optimal control problem can be converted into a stochastic target problem. Some technical results are delegated to the Appendix. Our main results are Theorems 3.1, 4.1, 5.1 and 6.1.

2 The Setup

To introduce the stochastic target problem in (3), we need to introduce some notation and make appropriate assumptions. Throughout this paper, the superscript $^\top$ stands for transposition, $|\cdot|$ for the Euclidean norm of a vector in \mathbb{R}^n and $\|\cdot\|$ for the Frobenius norm of a matrix. For a subset of \mathcal{O} of \mathbb{R}^n , we denote by $\text{Int}(\mathcal{O})$ its interior. We also denote the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$ by $B_r(x)$ and the set of $n \times n$ matrices by \mathbb{M}^n . Inequalities and inclusion between random variables and random sets, respectively, are in the almost sure sense unless otherwise stated.

Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{\lambda_i(\cdot, de)\}_{i=1}^I$ be a collection of independent integer-valued E -marked right-continuous point processes defined on this space. Here, E is a Borel subset of \mathbb{R} equipped with the Borel sigma field \mathcal{E} . Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_I)^\top$ and $W = \{W_s\}_{0 \leq s \leq T}$ be a d -dimensional Brownian motion defined on the same probability space such that W and λ are independent. Given $t \in [0, T]$, let $\mathbb{F}^t = \{\mathcal{F}_s^t, t \leq s \leq T\}$ be \mathbb{P} -completed filtration generated by $W \cdot -W_t$ and $\lambda([0, \cdot], de) - \lambda([0, t], de)$. Set $\mathcal{F}_s^t = \mathcal{F}_t^t$ for $0 \leq s < t$. We will use \mathcal{T}_t to denote the set of \mathbb{F}^t -stopping times valued in $[t, T]$. Given $\tau \in \mathcal{T}_t$, the set of \mathbb{F}^t -stopping times valued in $[\tau, T]$ will be denoted by \mathcal{T}_τ .

Assumption 2.1 λ satisfies the following:

1. $\lambda(ds, de)$ has intensity kernel $m(de)ds$ such that m_i is a Borel measure on (E, \mathcal{E}) for any $i = 1, \dots, I$ and $\hat{m}(E) < \infty$, where $m = (m_1, \dots, m_I)^\top$ and $\hat{m} = \sum_{i=1}^I m_i$.
2. $E = \text{supp}(m_i)$ for all $i = 1, 2, \dots, I$. Here, $\text{supp}(m_i) := \{e \in E : e \in N_e \in T_E \implies m_i(N_e) > 0\}$, where T_E is the topology on E induced by the Euclidean topology.
3. There exists a constant $C > 0$ such that

$$\mathbb{P} \left(\left\{ \hat{\lambda}(\{s\}, E) \leq C \text{ for all } s \in [0, T] \right\} \right) = 1, \text{ where } \hat{\lambda} = \sum_{i=1}^I \lambda_i.$$

The above assumption implies that there are a finite number of jumps during any finite time interval. Let $\tilde{\lambda}(ds, de) := \lambda(ds, de) - m(de)ds$ be the associated compensated random measure.

Let \mathcal{U}_1^t be the collection of all the \mathbb{F}^t -predictable processes in $\mathbb{L}^2(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}[0, T], \mathbb{P} \otimes \lambda_L; U_1)$, where λ_L is the Lebesgue measure on \mathbb{R} and $U_1 \subset \mathbb{R}^q$ for some $q \in \mathbb{N}$. Define \mathcal{U}_2^t to be the collection of all the maps $\nu_2 : \Omega \times [0, T] \times E \rightarrow \mathbb{R}^n$ which are $\mathcal{P}^t \otimes \mathcal{E}$ measurable such that

$$\|\nu_2\|_{\mathcal{U}_2^t} := \left(\mathbb{E} \left[\int_t^T \int_E |\nu_2(s, e)|^2 \hat{m}(de) ds \right] \right)^{\frac{1}{2}} < \infty,$$

where \mathcal{P}^t is the \mathbb{F}^t -predictable sigma-algebra on $\Omega \times [0, T]$. $\nu = (\nu_1, \nu_2) \in \mathcal{U}_0^t := \mathcal{U}_1^t \times \mathcal{U}_2^t$ takes value in the set $U := U_1 \times \mathbb{L}^2(E, \mathcal{E}, \hat{m}; \mathbb{R}^n)$. Let $\mathbb{D} = [0, T] \times \mathbb{R}^d$, $\mathbb{D}_i = [0, T[\times \mathbb{R}^d$ and $\mathbb{D}_T = \{T\} \times \mathbb{R}^d$. Given $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$, $t \in [0, T]$ and $\nu \in \mathcal{U}_0^t$, we consider the stochastic differential equations (SDEs)

$$\begin{aligned} dX(s) &= \mu_X(s, X(s), \nu(s))ds + \sigma_X(s, X(s), \nu(s))dW_s + \int_E \beta(s, X(s-), \nu_1(s), \nu_2(s, e), e)\lambda(ds, de), \\ dY(s) &= \mu_Y(s, Z(s), \nu(s))ds + \sigma_Y^\top(s, Z(s), \nu(s))dW_s + \int_E b^\top(s, Z(s-), \nu_1(s), \nu_2(s, e), e)\lambda(ds, de), \end{aligned} \tag{1}$$

with $(X(t), Y(t)) = (x, y)$. Here, $Z = (X, Y)$. In (1),

$$\begin{aligned} \mu_X : \mathbb{D} \times U &\rightarrow \mathbb{R}^d, \quad \sigma_X : \mathbb{D} \times U \rightarrow \mathbb{R}^{d \times d}, \quad \beta : \mathbb{D} \times U_1 \times \mathbb{R}^n \times E \rightarrow \mathbb{R}^{d \times I}, \\ \mu_Y : \mathbb{D} \times \mathbb{R} \times U &\rightarrow \mathbb{R}, \quad \sigma_Y : \mathbb{D} \times \mathbb{R} \times U \rightarrow \mathbb{R}^d, \quad b : \mathbb{D} \times \mathbb{R} \times U_1 \times \mathbb{R}^n \times E \rightarrow \mathbb{R}^I. \end{aligned}$$

Besides the measurability and the integrability conditions for \mathcal{U}_0^t , we impose another condition on the admissible control set. Let \mathcal{U}^t be the admissible control set, which consists of all $\nu \in \mathcal{U}_0^t$ such that for any compact set $C \subset \mathbb{R}^d \times \mathbb{R}$, there exists a constant $K_{C,\nu} > 0$ such that

$$\left| \int_E b^\top(\tau, x, y, \nu_1(\tau), \nu_2(\tau, e), e) \lambda(\{\tau\}, e) \right| \leq K_{C,\nu} \text{ for all } (x, y) \in C \text{ and } \tau \in \mathcal{T}_t. \quad (2)$$

Assumption 2.2 Let $z = (x, y)$ and $u = (u_1, u_2) \in U = U_1 \times \mathbb{L}^2(E, \mathcal{E}, \hat{m}; \mathbb{R}^n)$. We use the notation $\|u\|_U := |u_1| + \|u_2\|_{\hat{m}}$ and $u(e) := (u_1, u_2(e))$ for the rest of the paper.

1. μ_X, σ_X, μ_Y and σ_Y are all continuous;
2. $\mu_X, \sigma_X, \mu_Y, \sigma_Y$ are Lipschitz in z and locally Lipschitz in other variables. In addition,

$$|\mu_X(t, x, u)| + |\sigma_X(t, x, u)| \leq L(1 + |x| + \|u\|_U), \quad |\mu_Y(t, x, y, u)| + |\sigma_Y(t, x, y, u)| \leq L(1 + |y| + \|u\|_U).$$

3. b and β are Lipschitz and grow linearly in all variables except e , but uniformly in e .

Remark 2.1 Assumptions 2.1 and 2.2 guarantee that there exists a unique strong solution $(X_{t,x}^\nu, Y_{t,x,y}^\nu)$ to (1) for any $\nu \in \mathcal{U}^t$. Moreover, the processes $(X_{t,x}^\nu, Y_{t,x,y}^\nu)$ are càdlàg.

Remark 2.2 Under Assumptions 2.1 and 2.2, \mathcal{U}^t contains all the bounded processes in \mathcal{U}_0^t .¹

We now define the value function of the stochastic target problem. Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function with polynomial growth. The value function of the target problem is defined by

$$u(t, x) := \inf \left\{ y : \exists \nu \in \mathcal{U}^t \text{ s.t. } Y_{t,x,y}^\nu(T) \geq g(X_{t,x}^\nu(T)) \text{ } \mathbb{P} - \text{a.s.} \right\}. \quad (3)$$

2.1 The Hamilton-Jacobi-Bellman Equation

Denote $b = (b_1, b_2, \dots, b_I)^\top$ and $\beta = (\beta_1, \beta_2, \dots, \beta_I)$. For a given $\varphi \in C(\mathbb{D})$, we define the relaxed semi-limits

$$H^*(\Theta, \varphi) := \limsup_{\substack{\varepsilon \searrow 0, \Theta' \rightarrow \Theta \\ \eta \searrow 0, \psi \xrightarrow{\text{u.c.}} \varphi}} H_{\varepsilon, \eta}(\Theta', \psi) \text{ and } H_*(\Theta, \varphi) := \liminf_{\substack{\varepsilon \searrow 0, \Theta' \rightarrow \Theta \\ \eta \searrow 0, \psi \xrightarrow{\text{u.c.}} \varphi}} H_{\varepsilon, \eta}(\Theta', \psi).^2 \quad (4)$$

¹ The bound may depend on the process.

² The convergence $\psi \xrightarrow{\text{u.c.}} \varphi$ is understood in the sense that ψ converges uniformly on compact subsets to φ .

Here, for $\Theta = (t, x, y, p, A) \in \mathbb{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d$, $\varphi \in C(\mathbb{D})$, $\varepsilon \geq 0$ and $\eta \in [-1, 1]$,

$$H_{\varepsilon, \eta}(\Theta, \varphi) := \sup_{u \in \mathcal{N}_{\varepsilon, \eta}(t, x, y, p, \varphi)} \mathbf{F}^u(\Theta), \text{ where,}$$

$$\mathbf{F}^u(\Theta) := \mu_Y(t, x, y, u) - \mu_X^\top(t, x, u)p - \frac{1}{2} \text{Tr}[\sigma_X \sigma_X^\top(t, x, u)A], \quad N^u(t, x, y, p) := \sigma_Y(t, x, y, u) - \sigma_X^\top(t, x, u)p,$$

$$\Delta^{u, e}(t, x, y, \varphi) := \min_{1 \leq i \leq I} \{b_i(t, x, y, u(e), e) - \varphi(t, x + \beta_i(t, x, u(e), e)) + \varphi(t, x)\},$$

$$\mathcal{N}_{\varepsilon, \eta}(t, x, y, p, \varphi) := \{u \in U : |N^u(t, x, y, p)| \leq \varepsilon \text{ and } \Delta^{u, e}(t, x, y, \varphi) \geq \eta \text{ for } \hat{m} - \text{a.s. } e \in E\}.$$

For our later use, we also define the following:

$$J_i^{u, e}(t, x, y, \varphi) := b_i(t, x, y, u(e), e) - \varphi(t, x + \beta_i(t, x, u(e), e)) + \varphi(t, x),$$

$$\overline{J}^{u, e}(t, x, y, \varphi) := (J_1^{u, e}(t, x, y, \varphi), \dots, J_I^{u, e}(t, x, y, \varphi))^\top, \quad J^u(t, x, y, \varphi) := \inf_{e \in E} \min_{1 \leq i \leq I} J_i^{u, e}(t, x, y, \varphi),$$

$$\mathcal{L}^u \varphi(t, x) := \varphi_t(t, x) + \mu_X^\top(t, x, u) D \varphi(t, x) + \frac{1}{2} \text{Tr}[\sigma_X \sigma_X^\top(t, x, u) D^2 \varphi(t, x)].$$

Remark 2.3 For simplicity, we denote $H^*(t, x, \varphi(t, x), D\varphi(t, x), D^2\varphi(t, x), \varphi)$ by $H^*\varphi(t, x)$ for $\varphi \in C^{1,2}(\mathbb{D})$.

For $\varphi \in C^2(\mathbb{R}^d)$, we denote $H^*(T, x, \varphi(x), D\varphi(x), D^2\varphi(x), \varphi)$ by $H^*\varphi(x)$. We will use similar notation for H_* and other operators in later sections.

Later, we will produce a viscosity super-solution and sub-solution, respectively, to

$$-\partial_t \varphi(t, x) + H^* \varphi(t, x) \geq 0 \text{ in } \mathbb{D}_i \text{ and} \quad (5)$$

$$-\partial_t \varphi(t, x) + H_* \varphi(t, x) \leq 0 \text{ in } \mathbb{D}_i. \quad (6)$$

2.2 Stochastic Semi-Solutions

Before we introduce the definitions of the stochastic semi-solutions, we define the concatenation of the admissible controls.

Definition 2.1 (Concatenation) Let $\nu_1, \nu_2 \in \mathcal{U}^t$, $\tau \in \mathcal{T}_t$. The concatenation of ν_1 and ν_2 at τ is defined

as $\nu_1 \otimes_\tau \nu_2 := \nu_1 \mathbb{1}_{[0, \tau[} + \nu_2 \mathbb{1}_{[\tau, T]} \in \mathcal{U}^t$.³

³ This can be easily checked.

Definition 2.2 (Stochastic Super-solutions) A continuous function $w : \mathbb{D} \rightarrow \mathbb{R}$ is called a stochastic super-solution if

1. $w(T, x) \geq g(x)$ and for some $C > 0$ and $n \in \mathbb{N}$,⁴ $|w(t, x)| \leq C(1 + |x|^n)$ for all $(t, x) \in \mathbb{D}$.
2. Given $(t, x, y) \in \mathbb{D} \times \mathbb{R}$, for any $\tau \in \mathcal{T}_t$ and $\nu \in \mathcal{U}^t$, there exists $\tilde{\nu} \in \mathcal{U}^t$ such that $Y(\rho) \geq w(\rho, X(\rho))$ \mathbb{P} -a.s. on $\{Y(\tau) \geq w(\tau, X(\tau))\}$ for all $\rho \in \mathcal{T}_\tau$, where $X := X_{t,x}^{\nu \otimes \tau \tilde{\nu}}$ and $Y := Y_{t,x,y}^{\nu \otimes \tau \tilde{\nu}}$.

Definition 2.3 (Stochastic Sub-solutions) A continuous function $w : \mathbb{D} \rightarrow \mathbb{R}$ is called a stochastic sub-solution if

1. $w(T, x) \leq g(x)$ and for some $C > 0$ and $n \in \mathbb{N}$, $|w(t, x)| \leq C(1 + |x|^n)$ for all $(t, x) \in \mathbb{D}$.
2. Given $(t, x, y) \in \mathbb{D} \times \mathbb{R}$, for any $\tau \in \mathcal{T}_t$ and $\nu \in \mathcal{U}^t$, we have $\mathbb{P}(Y(\rho) < w(\rho, X(\rho)) | B) > 0$ for all $\rho \in \mathcal{T}_\tau$ and $B \subset \{Y(\tau) < w(\tau, X(\tau))\}$ satisfying $B \in \mathcal{F}_\tau^t$ and $\mathbb{P}(B) > 0$. Here, we use the notation $X := X_{t,x}^\nu$ and $Y := Y_{t,x,y}^\nu$.

Denote the sets of stochastic super-solutions and sub-solutions by \mathbb{U}^+ and \mathbb{U}^- , respectively.

Assumption 2.3 \mathbb{U}^+ and \mathbb{U}^- are not empty.

Remark 2.4 Let $u^+ := \inf_{w \in \mathbb{U}^+} w$. For any stochastic super-solution w , choose $\tau = t$ and $\rho = T$. Then there exists $\tilde{\nu} \in \mathcal{U}^t$ such that $Y_{t,x,y}^{\tilde{\nu}}(T) \geq w(T, X_{t,x}^{\tilde{\nu}}(T)) \geq g(X_{t,x}^{\tilde{\nu}}(T))$ \mathbb{P} -a.s. if $y \geq w(t, x)$. Hence, $y \geq w(t, x)$ implies that $y \geq u(t, x)$ from (3). This means that $w \geq u$ and $u^+ \geq u$. By the definition of \mathbb{U}^+ , we know that $u^+(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.5 Let $u^- := \sup_{w \in \mathbb{U}^-} w$. For any stochastic sub-solution w , if $y < w(t, x)$, by choosing $\tau = t$ and $\rho = T$, we get that for any $\nu \in \mathcal{U}^t$, $\mathbb{P}(Y_{t,x,y}^\nu(T) < g(X_{t,x}^\nu(T))) \geq \mathbb{P}(Y_{t,x,y}^\nu(T) < w(T, X_{t,x}^\nu(T))) > 0$. Therefore, from (3), $y < w(t, x)$ implies that $y \leq u(t, x)$. This means that $w \leq u$ and $u^- \leq u$. By the definition of \mathbb{U}^- , it holds that $u^-(T, x) \leq g(x)$ for all $x \in \mathbb{R}^d$.

In short,

$$u^- = \sup_{w \in \mathbb{U}^-} w \leq u \leq \inf_{w \in \mathbb{U}^+} w = u^+. \quad (7)$$

We will provide sufficient conditions which guarantee Assumption 2.3 in the Appendix A. As in [4] and [5], the proof of the sub-solution property requires a regularity assumption on the set-valued map $\mathcal{N}_{0,\eta}(\cdot, \psi)$.

⁴ C and N may depend on w and T . This also applies to Definition 2.3

Assumption 2.4 For $\psi \in C(\mathbb{D})$, $\eta > 0$, let B be a subset of $\mathbb{D} \times \mathbb{R} \times \mathbb{R}^d$ such that $\mathcal{N}_{0,\eta}(\cdot, \psi) \neq \emptyset$ on B . Then for every $\varepsilon > 0$, $(t_0, x_0, y_0, p_0) \in \text{Int}(B)$ and $u_0 \in \mathcal{N}_{0,\eta}(t_0, x_0, y_0, p_0, \psi)$, there exists an open neighborhood B' of (t_0, x_0, y_0, p_0) and a locally Lipschitz continuous map \hat{v} defined on B' such that $\|\hat{v}(t_0, x_0, y_0, p_0) - u_0\|_U \leq \varepsilon$ and $\hat{v}(t, x, y, p) \in \mathcal{N}_{0,\eta}(t, x, y, p, \psi)$.

3 Viscosity Property in \mathbb{D}_i

In this section, we state and prove the theorem which characterizes u^+ (resp. u^-) as a viscosity sub-solution (resp. super-solution) of (6) (resp. (5)). The boundary conditions will be discussed in Theorem 4.1.

Lemma 3.1 \mathbb{U}^+ and \mathbb{U}^- are closed under pairwise minimization and maximization, respectively. That is,

1. if $w_1, w_2 \in \mathbb{U}^+$, then $w_1 \wedge w_2 \in \mathbb{U}^+$; 2. if $w_1, w_2 \in \mathbb{U}^-$, then $w_1 \vee w_2 \in \mathbb{U}^-$.

Lemma 3.2 There exists a non-increasing sequence $\{w_n\}_{n=1}^\infty \subset \mathbb{U}^+$ such that $w_n \searrow u^+$ and a non-increasing sequence $\{v_n\}_{n=1}^\infty \subset \mathbb{U}^-$ such that $v_n \nearrow u^-$.

Theorem 3.1 Under Assumptions 2.1-2.4, u^+ is an upper semi-continuous (USC) viscosity sub-solution of (6). On the other hand, under Assumptions 2.1-2.3, u^- is a lower semi-continuous (LSC) viscosity super-solution of (5).

Proof See Appendix B. □

4 Boundary Conditions

In this section, we discuss the boundary conditions at T . From the definition of the value function u , it holds that $u(T, x) = g(x)$ for all $x \in \mathbb{R}^d$. However, u^+ and u^- may not satisfy this boundary condition. Define

$$\mathbf{N}(t, x, y, p, \psi) := \{(r, s) \in \mathbb{R}^d \times \mathbb{R} : \exists u \in U, \text{ s.t. } r = N^u(t, x, y, p) \text{ and } s \leq \Delta^{u,e}(t, x, y, \psi) \hat{m} - \text{a.s.}\}$$

and $\delta := \text{dist}(0, \mathbf{N}^c) - \text{dist}(0, \mathbf{N})$, where dist denotes the Euclidean distance. It holds that

$$0 \in \text{int}(N(t, x, y, p, \psi)) \text{ iff } \delta(t, x, y, p, \psi) > 0. \quad (8)$$

The upper (resp. lower) semi-continuous envelope of δ is denoted by δ^* (resp. δ_*). Let

$$u^+(T-, x) = \limsup_{(t < T, x') \rightarrow (T, x)} u^-(t, x'), \quad u^-(T-, x) = \liminf_{(t < T, x') \rightarrow (T, x)} u^-(t, x').$$

The following theorem is an adaptation of the results in [2, 3, 4, 11].

Theorem 4.1 *Under Assumptions 2.1-2.4, if g is USC, then $u^+(T-, \cdot)$ is a USC viscosity sub-solution of $\min\{\varphi(x) - g(x), \delta_*\varphi(x)\} \leq 0$ on \mathbb{R}^d . On the other hand, under Assumptions 2.1-2.3, if g is LSC, $u^-(T-, \cdot)$ is an LSC viscosity super-solution of $\min\{(\varphi(x) - g(x))\mathbb{1}_{\{H^*\varphi(x) < \infty\}}, \delta^*\varphi(x)\} \geq 0$ on \mathbb{R}^d .*

Proof Step 1 (The sub-solution property on \mathbb{D}_T). For the sake of contradiction, we assume that for some $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ satisfying $0 = u^+(T-, x_0) - \varphi(x_0) = \max_{x \in \mathbb{R}^d} (u^+(T-, x) - \varphi(x))$, it holds that $\varphi(x_0) - g(x_0) > 2\eta$ and $\delta_*\varphi(x_0) > 2\eta$ for some $\eta > 0$. Let $\{w_k\}_{k=1}^\infty$ be a sequence in \mathbb{U}^+ such that $w_k \searrow u^+$. Set $\tilde{\varphi}(t, x) = \varphi(x) + \iota|x - x_0|^{n_0} + \iota\sqrt{T-t}$ for $\iota > 0$, where ι will be fixed later and n_0 satisfies

$$\min_{0 \leq t \leq T} (\tilde{\varphi}(t, x) - w_1(t, x)) \rightarrow \infty \text{ as } |x| \rightarrow \infty \text{ for any } \iota > 0.$$

By the lower semi-continuity of δ_* and the upper semi-continuity of g , we can find $\iota > 0$ and $\varepsilon > 0$ such that

$$\tilde{\varphi}(t, x) - g(x) > \eta \text{ and} \tag{9}$$

$$\delta_*(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi}) \geq \eta \text{ for } (t, x) \in [T - \varepsilon, T] \times \text{cl}(B_\varepsilon(x_0)) \text{ and } |y - \tilde{\varphi}(t, x)| \leq \varepsilon. \tag{10}$$

By Assumption 2.4, the fact that $\delta \geq \delta_*$, (8) and (10), we can find a locally Lipschitz map $\hat{\nu}$ such that

$$\hat{\nu}(t, x, y, D\tilde{\varphi}(t, x)) \in \mathcal{N}_{0, \eta}(t, x, y, \tilde{\varphi}(t, x), \tilde{\varphi}) \tag{11}$$

for all $(t, x, y) \in \mathbb{D} \times R$ s.t. $(t, x) \in [T - \varepsilon, T] \times \text{cl}(B_\varepsilon(x_0))$ and $|y - \tilde{\varphi}(t, x)| \leq \varepsilon$.

In (11), we may need to choose smaller values of ε, ι and η . Fix ι . Since $\partial_t \tilde{\varphi}(t, x) \rightarrow -\infty$ as $t \rightarrow T$, by the continuity of μ_Y, μ_X, σ_X and ν ,

$$\mu_Y(t, x, y, \hat{\nu}(t, x, y, D\tilde{\varphi}(t, x))) - \mathcal{L}^{\hat{\nu}(t, x, y, D\tilde{\varphi}(t, x))} \tilde{\varphi}(t, x) \geq \eta, \tag{12}$$

for all $(t, x, y) \in \mathbb{D} \times R$ s.t. $(t, x) \in [T - \varepsilon, T] \times \text{cl}(B_\varepsilon(x_0))$ and $|y - \tilde{\varphi}(t, x)| \leq \varepsilon$.

Here we may need to shrink $\varepsilon > 0$ again. Since u^+ is USC and $\tilde{\varphi}(T, x_0) = u^+(T-, x_0)$, there exists $\alpha > 0$ such that $\tilde{\varphi} > u^+ - 2\alpha$ on $[T - \varepsilon, T[\times \text{cl}(B_{\varepsilon/2}(x_0))$ after possibly shrinking ε another time. Since $w_k \searrow u^+$, there exists $n_0 \in \mathbb{N}$ such that

$$\tilde{\varphi} > w_{n_0} - \alpha \text{ on } [T - \varepsilon, T[\times \text{cl}(B_{\varepsilon/2}(x_0)). \quad (13)$$

Since $\min_{0 \leq t \leq T} (\tilde{\varphi}(t, x) - w_1(t, x)) \rightarrow \infty$ as $|x| \rightarrow \infty$, we can find $R_0 > \varepsilon$ such that

$$\tilde{\varphi} > w_{n_0} + \varepsilon \text{ on } \mathbb{O} := [T - \varepsilon, T] \times (\mathbb{R}^d \setminus \text{cl}(B_{R_0}(x_0))). \quad (14)$$

Notice that $\tilde{\varphi}(T, \cdot) - u^+(T-, \cdot)$ is strictly positive on the compact set $\mathbb{T}^* := \text{cl}(B_{R_0}(x_0)) - B_{\varepsilon/2}(x_0)$. Hence, by the upper semi-continuity of $u^+(T-, \cdot)$, there exists $\zeta > 0$ such that

$$\tilde{\varphi}(T, \cdot) > u^+(T-, \cdot) + 4\zeta \text{ on } \mathbb{T}^*. \quad (15)$$

From (15), we conclude that there exists $\sigma > 0$ such that

$$\tilde{\varphi} > u^+ + 2\zeta \text{ on } [T - \sigma, T[\times \mathbb{T}^*. \quad (16)$$

More precisely, if (16) does not hold for any $\sigma > 0$, then there exists a sequence $(t_n, x_n) \in \mathbb{D}_i$ such that $t_n \rightarrow T$, $x_n \in \mathbb{T}^*$ and $\tilde{\varphi}(t_n, x_n) \leq u^+(t_n, x_n) + 2\zeta$. The compactness of \mathbb{T}^* implies that there is a subsequence of (t_n, x_n) which converges to (T, x') for some $x' \in \mathbb{T}^*$. By taking the limsup of the above equation over the subsequence, we get $\tilde{\varphi}(T, x') \leq u^+(T-, x') + 2\zeta$. This contradicts (15). Therefore, (16) holds.

In (16), we choose $\sigma < \varepsilon$. By a Dini-type argument, there exists $n_1 \geq n_0$ such that

$$\tilde{\varphi} > w_{n_1} + \zeta \text{ on } [T - \sigma, T[\times \mathbb{T}^*. \quad (17)$$

Set $w = w_{n_1}$. For $\kappa \in]0, \varepsilon \wedge \alpha \wedge \zeta[$, define

$$w^\kappa := \begin{cases} (\tilde{\varphi} - \kappa) \wedge w & \text{on } [T - \sigma, T] \times \text{cl}(B_\varepsilon(x_0)), \\ w & \text{outside } [T - \sigma, T] \times \text{cl}(B_\varepsilon(x_0)). \end{cases}$$

Since $w(T, x) \geq g(x)$ and (9) holds, we get that $w^\kappa(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$. We also notice that

$$w^\kappa(T, x_0) \leq \varphi(x_0) - \kappa < u^+(T-, x_0) \leq u^+(T, x_0). \quad (18)$$

Using (11), (12), (13), (14) and (17) in a manner that is similar to Step 1 in Theorem 3.1's proof, we can show that w^κ is a stochastic super-solution, which contradicts (18).

Step 2 (The super-solution property on \mathbb{D}_T). We will divide the proof into two steps:

Step 2.A. We will show that $u^-(T-, \cdot)$ is a viscosity super-solution of $(\varphi(x) - g(x)) \mathbb{1}_{\{H^*\varphi(x) < \infty\}} \geq 0$ on \mathbb{R}^d .

Let $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ be such that $0 = (u^-(T-, x_0) - \varphi(x_0)) = \min_{x \in \mathbb{R}^d} (u^-(T-, x) - \varphi(x))$. Assuming that $H^*\varphi(x_0) = C < \infty$ and that $g(x_0) > u^-(T-, x_0) = \varphi(x_0)$, we will work towards a contradiction. Let $\{w_k\}_1^\infty$ be a sequence in \mathbb{U}^- such that $w_n \nearrow u^-$. Let $\tilde{\varphi}(t, x) = \varphi(x) - \iota|x - x_0|^{n_0} - (C + 2)(T - t)$ and $\tilde{\varphi}'(x) = \varphi(x) - \iota|x - x_0|^{n_0}$ for $\iota > 0$, where ι will be fixed later and $n_0 \geq 2$ satisfies

$$\max_{0 \leq t \leq T} (\tilde{\varphi}(t, x) - w_1(t, x)) \rightarrow -\infty \text{ and } \max_{0 \leq t \leq T} \tilde{\varphi}(t, x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty \text{ for any } \iota > 0. \quad (19)$$

Note that $D\tilde{\varphi}'(x) = D\tilde{\varphi}(t, x)$ and $D^2\tilde{\varphi}'(x) = D^2\tilde{\varphi}(t, x)$. From $g(x_0) > \varphi(x_0) = \tilde{\varphi}(T, x_0) = u^-(T-, x_0)$ and the lower semi-continuity of g and u^- , we can find $\varepsilon > 0$ and $\eta \in]0, 1[$ such that

$$g(x) - \tilde{\varphi}(t, x) > \varepsilon \text{ for } (t, x) \in \text{cl}(B_\varepsilon(T, x_0)), \quad \tilde{\varphi} < u^- + 2\eta \text{ on } [T - \varepsilon, T] \times \text{cl}(B_{\varepsilon/2}(x_0)). \quad (20)$$

By the locally boundedness of μ_X , σ_X , μ_Y , b and β , and $H^*\varphi(x_0) = C$, there exists $\iota > 0$ such that

$$\mu_Y(t, x, y, u) - \mu_X^\top(t, x, u)D\tilde{\varphi}(t, x) - \frac{1}{2}\text{Tr}[\sigma_X\sigma_X^\top(t, x, u)D^2\tilde{\varphi}(t, x)] \leq C + 1 \text{ for all } (t, x, y, u) \in \mathbb{D} \times \mathbb{R} \times U$$

satisfying $(t, x) \in [T - \varepsilon, T] \times \text{cl}(B_\varepsilon(x_0))$, $|y - \tilde{\varphi}(t, x)| \leq \varepsilon$ and $u \in \mathcal{N}_{\varepsilon, -\eta}(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi}')$. Here, we may need to choose smaller values of ε and η . Therefore, by the definition of $\Delta^{u, e}$,

$$\begin{aligned} \mu_Y(t, x, y, u) - \mathcal{L}^u \tilde{\varphi}(t, x) &\leq C + 1 - C - 2 \leq -\eta \text{ for all } (t, x, y) \in \mathbb{D} \times \mathbb{R} \times U \\ \text{s.t. } (t, x) &\in [T - \varepsilon, T] \times \text{cl}(B_\varepsilon(x_0)), |y - \tilde{\varphi}(t, x)| \leq \varepsilon \text{ and } u \in \mathcal{N}_{\varepsilon, -\eta}(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi}). \end{aligned}$$

Fix ι . Since $w_k \nearrow u^-$, there exists $n_0 \in \mathbb{N}$ such that $\tilde{\varphi} < w_{n_0} + \eta$ on $[T - \varepsilon, T] \times \text{cl}(B_{\varepsilon/2}(x_0))$ due to (20). By (19), there exists $R_0 > \varepsilon$ such that $\tilde{\varphi}(t, x) < w_{n_0}(t, x) + \varepsilon \leq w_n(t, x) + \varepsilon$ on \mathbb{O} for $n \geq n_0$, where $\mathbb{O} := [T - \varepsilon, T] \times (\mathbb{R}^d \setminus \text{cl}(B_{R_0}(x_0)))$. Since $\tilde{\varphi}(T, x) \leq \varphi(x)$, $u^-(T-, \cdot) - \tilde{\varphi}(T, \cdot)$ is strictly positive on the compact set $\mathbb{T}^* := \text{cl}(B_{R_0}(x_0)) - B_{\varepsilon/2}(x_0)$. Hence, by the lower semi-continuity of u^- , there exists $\alpha > 0$ such that $\tilde{\varphi}(T, \cdot) < u^-(T-, \cdot) - 4\alpha$ on \mathbb{T}^* . Similar to Step 1 in this proof, we can find $\sigma \in]0, \varepsilon[$ and $n_1 \geq n_0$ such that $\tilde{\varphi} < w_{n_1} - \alpha$ on $[T - \sigma, T] \times \mathbb{T}^*$. Set $w = w_{n_1}$. For $\kappa \in]0, \varepsilon \wedge \delta \wedge \alpha[$, define

$$w^\kappa := \begin{cases} (\tilde{\varphi} + \kappa) \vee w & \text{on } [T - \sigma, T] \times \text{cl}(B_\varepsilon(x_0)), \\ w & \text{outside } [T - \sigma, T] \times \text{cl}(B_\varepsilon(x_0)). \end{cases}$$

As in Step 2 of Theorem 3.1's proof, we can show that $w^\kappa \in \mathbb{U}^-$, which yields a contradiction.

Step 2.B: In this step, we prove that $u^-(T-, \cdot)$ is a viscosity super-solution of $\delta^* \varphi(x) \geq 0$. Let $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ be such that $0 = (u^-(T-, x_0) - \varphi(x_0)) = \min_{\mathbb{R}^d} (u^-(T-, x) - \varphi(x))$. Let (s_n, ξ_n) be a sequence in \mathbb{D}_i satisfying $(s_n, \xi_n) \rightarrow (T, x_0)$ and $u^-(s_n, \xi_n) \rightarrow u^-(T-, x_0) = \varphi(x_0)$. For all $n \in \mathbb{N}$, $k \geq 0$ and $\iota \geq 0$, define

$$\varphi_n^{k, \iota}(t, x) = \varphi(x) - \iota |x - x_0|^4 + k \frac{T - t}{(T - s_n)}, \quad \varphi^\iota(x) = \varphi(x) - \iota |x - x_0|^4.$$

Notice that

$$\lim_{\iota \rightarrow 0} \lim_{k \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{(t, x) \in [s_n, T] \times \text{cl}(B_1(x_0))} |\varphi_n^{k, \iota}(t, x) - \varphi(x)| = 0.$$

Let $(t_n^{k, \iota}, x_n^{k, \iota})$ be the minimizer of $u^- - \varphi_n^{k, \iota}$ on $[s_n, T] \times \text{cl}(B_1(x_0))$. We claim that for any $k > 0$ and $\iota > 0$, there exists $N^{k, \iota} \in \mathbb{N}$ such that

$$s_n \leq t_n^{k, \iota} < T \text{ for all } n \geq N^{k, \iota}, \text{ and } x_n^{k, \iota} \rightarrow x_0 \text{ as } n \rightarrow \infty. \quad (21)$$

We now prove (21). Since $(s_n, \xi_n) \rightarrow (T, x_0)$, we can find $N^{k,\iota} \in \mathbb{N}$ such that for $n \geq N^{k,\iota}$,

$$(u^- - \varphi_n^{k,\iota})(s_n, \xi_n) = u^-(s_n, \xi_n) - \varphi(\xi_n) + \iota|\xi_n - x_0|^4 - \frac{1}{k} \leq -\frac{1}{2k} < 0. \quad (22)$$

On the other hand,

$$\liminf_{t \uparrow T, x' \rightarrow x} (u^- - \varphi_n^{k,\iota})(t, x') = u^-(T-, x) - \varphi(x) + \iota|x - x_0|^4 \geq 0 \text{ for } |x - x_0| \leq 1. \quad (23)$$

By (22) and (23), the first part of (21) holds. By an argument similar to Step 4 in Theorem 3.1's proof in [9], we know that the second part of (21) also holds.

From (21) and the definition of $\varphi_n^{k,\iota}$, we also see that

$$\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}) \rightarrow u^-(T-, x_0) = \varphi(x_0) \text{ as } n \rightarrow \infty, \text{ then } k \rightarrow 0, \iota \rightarrow 0. \quad (24)$$

By (21), (24) and the facts that $u^-(t_n^{k,\iota}, x_n^{k,\iota}) \leq \varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota})$ and $\liminf_{t < T, x \rightarrow (T, x_0)} u^-(t, x) = u^-(T-, x_0)$, it holds that $u^-(t_n^{k,\iota}, x_n^{k,\iota}) \rightarrow u^-(T-, x_0) = \varphi(x_0)$ as $n \rightarrow \infty$ then $k \rightarrow 0, \iota \rightarrow 0$. Since for all $k > 0, \iota > 0$ and $n \geq N^{k,\iota}$, $(t_n^{k,\iota}, x_n^{k,\iota})$ is a local minimizer of $u^- - \varphi_n^{k,\iota}$ and $t_n^{k,\iota} < T$, we get

$$-\partial_t \varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}) + H^*(t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), D^2\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota})) \geq 0$$

from Theorem 3.1. For any $k > 0, \iota > 0$ and $n \geq N^{k,\iota}$, from the definition of H^* , there exists a sequence

$\{(\varepsilon_m, \eta_m, t_m, x_m, y_m, p_m, A_m, \varphi_m)\} \subset \mathbb{R}_+ \times [-1, 1] \times \mathbb{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d \times C(\mathbb{D})$ such that $(\varepsilon_m, \eta_m) \rightarrow (0, 0)$,

$$\varphi_m \xrightarrow{\text{u.c.}} \varphi_n^{k,\iota}(t_m, x_m, y_m, p_m, A_m) \rightarrow (t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), D^2\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota})) \text{ and } \quad (25)$$

$$H_{\varepsilon_m, \eta_m}(t_m, x_m, y_m, p_m, A_m, \varphi_m) \rightarrow H^*(t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), D^2\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota})) > -\infty.$$

This implies that $\mathcal{N}_{\varepsilon_m, \eta_m}(t_m, x_m, y_m, p_m, \varphi_m) \neq \emptyset$ since $\sup \emptyset = -\infty$. By the definition of δ , it holds that

$\delta(t_m, x_m, y_m, p_m, \varphi_m) \geq -\sqrt{\varepsilon_m^2 + \eta_m^2}$. From (25) and the definition of δ^* , we get

$$\delta^*(t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), \varphi_n^{k,\iota}) \geq \limsup_{m \rightarrow \infty} \delta(t_m, x_m, y_m, p_m, \varphi_m) \geq 0.$$

By the definition of $\Delta^{u,e}$ in the set-valued map \mathbf{N} , the equation above implies that

$$\delta^*(t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), \varphi^\iota) = \delta^*(t_n^{k,\iota}, x_n^{k,\iota}, u^-(t_n^{k,\iota}, x_n^{k,\iota}), D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}), \varphi_n^{k,\iota}) \geq 0. \quad (26)$$

Note that $\varphi^\iota \xrightarrow{\text{u.c.}} \varphi$ as $\iota \rightarrow 0$. Moreover, for $\iota > 0$, $u^-(t_n^{k,\iota}, x_n^{k,\iota}) \rightarrow \varphi(x_0)$ and $D\varphi_n^{k,\iota}(t_n^{k,\iota}, x_n^{k,\iota}) \rightarrow D\varphi(x_0)$ as $n \rightarrow \infty$ then $k \rightarrow 0$. Taking the limsup of (26) by first sending $n \rightarrow \infty$ then $k \rightarrow 0$ and $\iota \rightarrow 0$, we have $\delta^*\varphi(x_0) = \delta^*\varphi(T, x_0, \varphi(x_0), D\varphi(x_0), \varphi) \geq 0$ from the upper semi-continuity of δ^* , \square

5 Verification by Comparison

We now carry out the verification for non-smooth functions assuming the comparison principle as in [9].

Assumption 5.1 *Let $H = H_*$. Assume that $H = H^*$ on the set $\{H < \infty\}$ and that there exists an LSC function $G : \mathbb{D} \times \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d \times C(\mathbb{D}) \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} (a) \quad & H(t, x, y, p, A, \varphi) < \infty \implies G(t, x, y, p, A, \varphi) \leq 0, \\ (b) \quad & G(t, x, y, p, A, \varphi) < 0 \implies H(t, x, y, p, A, \varphi) < \infty. \end{aligned}$$

Proposition 5.1 *Under Assumptions 2.1-2.4 (resp. 2.1-2.3) and 5.1, u^+ (resp. u^-) is a USC (resp. an LSC) viscosity sub-solution (resp. super-solution) of $\max\{-\partial_t\varphi(t, x) + H\varphi(t, x), G\varphi(t, x)\} = 0$ on \mathbb{D}_i . Moreover, if g is USC, $u^+(T-, \cdot)$ is a USC viscosity sub-solution of $\min\{\max\{\varphi(x) - g(x), G\varphi(x)\}, \delta_*\varphi(x)\} \leq 0$ on \mathbb{R}^d . If g is LSC, $u^-(T-, \cdot)$ is an LSC viscosity super-solution of $\min\{\max\{\varphi(x) - g(x), G\varphi(x)\}, \delta^*\varphi(x)\} \geq 0$ on \mathbb{R}^d .*

Proof **(1) The sub-solution property in \mathbb{D}_i .** Suppose $0 = (u^+ - \varphi)(t_0, x_0) = \max_{\mathbb{D}_i}(u^+ - \varphi)$ for some $(t_0, x_0) \in \mathbb{D}_i$ and $\varphi \in C^{1,2}(\mathbb{D})$. Then $-\partial_t\varphi(t_0, x_0) + H\varphi(t_0, x_0) = -\partial_t\varphi(t_0, x_0) + H_*\varphi(t_0, x_0) \leq 0$ from Theorem 3.1. From (a) in Assumption 5.1, $G\varphi(t_0, x_0) \leq 0$. Therefore, the sub-solution property holds for u^+ in the parabolic interior.

(2) The super-solution property in \mathbb{D}_i . Suppose $0 = (u^- - \varphi)(t_0, x_0) = \min_{\mathbb{D}_i}(u^- - \varphi)$ for some $(t_0, x_0) \in \mathbb{D}_i$ and $\varphi \in C^{1,2}(\mathbb{D})$. If $H\varphi(t_0, x_0) < \infty$, $-\partial_t\varphi(t_0, x_0) + H\varphi(t_0, x_0) = -\partial_t\varphi(t_0, x_0) + H^*\varphi(t_0, x_0) \geq 0$ from Assumption 5.1 and Theorem 3.1. On the other hand, if $H\varphi(t_0, x_0) = \infty$, $G\varphi(t_0, x_0) \geq 0$ from (b) in Assumption 5.1. Therefore, the viscosity super-solution property holds for u^- in the parabolic interior.

(3) The sub-solution property on \mathbb{D}_T . From Theorem 4.1, we know that $u^+(T-, \cdot)$ is viscosity sub-solution of $\min\{\varphi(x) - g(x), \delta_* \varphi(x)\} \leq 0$. Therefore, it suffices to show that $Gu^+(T-, \cdot) \leq 0$ in the viscosity sense. Let $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ be such that $0 = (u^+(T-, x_0) - \varphi(x_0)) = \max_{x \in \mathbb{R}^d} (u^+(T-, x) - \varphi(x))$. Let (s_n, ξ_n) be a sequence in \mathbb{D}_i satisfying $(s_n, \xi_n) \rightarrow (T, x_0)$ and $u^+(s_n, \xi_n) \rightarrow u^+(T-, x_0)$. For all $n \in \mathbb{N}$, $k \geq 0$ and $\iota \geq 0$, define

$$\varphi_n^{k, \iota}(t, x) = \varphi(x) + \iota|x - x_0|^4 - k \frac{T - t}{(T - s_n)}, \varphi^\iota(x) = \varphi(x) + \iota|x - x_0|^4.$$

Let $(t_n^{k, \iota}, x_n^{k, \iota})$ be the maximizer of $u^+ - \varphi_n^{k, \iota}$ on $[s_n, T] \times \text{cl}(B_1(x_0))$. Similar to the arguments in Step 2B of Theorem 4.1's proof, we can show that $\lim_{k \rightarrow 0, \iota \rightarrow 0} \lim_{n \rightarrow \infty} u^+(t_n^{k, \iota}, x_n^{k, \iota}) = \varphi(x_0)$. We also know that for any $k > 0$ and $\iota > 0$, there exists $N^{k, \iota} \in \mathbb{N}$ such that $s_n \leq t_n^{k, \iota} < T$ for all $n \geq N^{k, \iota}$ and $x_n^{k, \iota} \rightarrow x_0$ as $n \rightarrow \infty$. Therefore, for all $k > 0$, $\iota > 0$ and $n \geq N^{k, \iota}$, $(t_n^{k, \iota}, x_n^{k, \iota})$ is a maximizer of $u^+ - \varphi_n^{k, \iota}$ on $[s_n, T] \times \text{cl}(B_1(x_0))$. From Theorem 3.1,

$$-\partial_t \varphi(t_n^{k, \iota}, x_n^{k, \iota}) + H_*(t_n^{k, \iota}, x_n^{k, \iota}, u^+(t_n^{k, \iota}, x_n^{k, \iota}), D\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), D^2\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), \varphi_n^{k, \iota}) \leq 0.$$

Hence, the H_* -term in the above equation is less than ∞ . From the definition of $\Delta^{u, e}$, we get

$$H_*(t_n^{k, \iota}, x_n^{k, \iota}, u^+(t_n^{k, \iota}, x_n^{k, \iota}), D\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), D^2\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), \varphi^\iota) < \infty, \text{ which further implies that}$$

$$G\varphi(t_n^{k, \iota}, x_n^{k, \iota}, u^+(t_n^{k, \iota}, x_n^{k, \iota}), D\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), D^2\varphi_n^{k, \iota}(t_n^{k, \iota}, x_n^{k, \iota}), \varphi^\iota) \leq 0 \text{ by Assumption 5.1.}$$

Using an argument similar to that in Step 2B of Theorem 4.1's proof, we conclude that $G\varphi(x_0) \leq 0$.

(4) The super-solution property on \mathbb{D}_T . It suffices to show that $u^-(T-, \cdot)$ is a viscosity super-solution of

$$\max\{\varphi(x) - g(x), G\varphi(x)\} \geq 0. \quad (27)$$

Let $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ be such that $0 = (u^-(T-, x_0) - \varphi(x_0)) = \min_{x \in \mathbb{R}^d} (u^-(T-, x) - \varphi(x))$. From Theorem 4.1, one of the following two scenarios must hold:

$$\varphi(x_0) \geq g(x_0), \quad H^*\varphi(x_0) < \infty \quad \text{or} \quad (28)$$

$$H^*\varphi(x_0) = \infty. \quad (29)$$

(28) implies (27); on the other hand, if (29) holds, then $H\varphi(x_0) = \infty$, which means that $G\varphi(x_0) \geq 0$ from (b) in Assumption 5.1. Therefore, (27) holds. \square

Assumption 5.2 *Assume that $\delta^* = \delta_*$, g is continuous and a comparison principle holds between USC sub-solutions and LSC super-solutions for*

$$\min\{\max\{\varphi(x) - g(x), G\varphi(x)\}, \delta\varphi(x)\} = 0 \quad \text{on } \mathbb{R}^d. \quad (30)$$

In the presence of jumps, it is nontrivial to check this assumption. When there are no jumps in the controlled processes, the comparison principle can be proved in certain classes of functions (see the discussion above Assumption 2.2 in [6]). Also, in Section 6, δ drops out in the corresponding PDE and there are comparison results available for fully non-linear equations with jumps (see [13]).

Lemma 5.1 *Under Assumptions 5.1, 5.2 and 2.1-2.4, $u^-(T-, \cdot) = u^+(T-, \cdot) = \hat{g}(\cdot)$, where \hat{g} is the unique continuous viscosity solution to (30).*

Proof It follows from their definitions that $u^- \leq u^+$. Since u^+ is USC and u^- is LSC, then

$$u^-(T-, x) = \liminf_{(t < T, x') \rightarrow (T, x)} u^-(t, x') \leq \limsup_{(t < T, x') \rightarrow (T, x)} u^+(t, x') = u^+(T-, x).$$

Moreover, $u^+(T-, \cdot)$ is a viscosity sub-solution and $u^-(T-, \cdot)$ is a viscosity super-solution to (30) due to Theorem 4.1. Therefore, the claim holds by Assumption 5.2. \square

Theorem 5.1 *Suppose that there is a comparison principle for*

$$\max\{-\partial_t \varphi(t, x) + H\varphi(t, x), G\varphi(t, x)\} = 0 \quad \text{on } \mathbb{D}_i \quad (31)$$

and that Assumptions 2.1-2.4, 5.1 and 5.2 hold. Then there exists a unique continuous viscosity solution V to (31) with terminal condition $V(T, \cdot) = \hat{g}(\cdot)$ and $u(t, x) = u^-(t, x) = u^+(t, x) = V(t, x)$ for $(t, x) \in \mathbb{D}_i$.

Proof Define

$$\hat{u}^+(t, x) := \begin{cases} u^+(t, x), & (t, x) \in \mathbb{D}_i \\ \hat{g}(x), & t = T, x \in \mathbb{R}^d \end{cases} \quad \text{and} \quad \hat{u}^-(t, x) := \begin{cases} u^-(t, x), & (t, x) \in \mathbb{D}_i, \\ \hat{g}(x), & t = T, x \in \mathbb{R}^d. \end{cases}$$

From Proposition 5.1, \hat{u}^- is an LSC viscosity super-solution and \hat{u}^+ is a USC viscosity sub-solution of (31). Since $\hat{u}^+(T, \cdot) = \hat{u}^-(T, \cdot)$, $\hat{u}^+ \leq \hat{u}^-$ on \mathbb{D} by comparison. Hence, $\hat{u}^+ = \hat{u}^-$ on \mathbb{D} from (7). Define $V := \hat{u}^+ = \hat{u}^-$. It is a continuous viscosity solution of (31) satisfying $V(T, x) = \hat{g}(x)$. Uniqueness follows directly from the comparison principle. \square

6 Stochastic Control as a Stochastic Target Problem

In this section, we show how the HJB equation associated to an optimal control problem in standard form can be deduced from a stochastic target problem. Given a bounded continuous function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, we define an optimal control problem by $\mathbf{u}(t, x) := \inf_{\nu \in \mathcal{U}^t} \mathbb{E}[g(X_{t,x}^\nu(T))]$. We follow the setup of Section 2 with one exception: \mathcal{U}^t is the collection of all the \mathbb{F}^t -predictable processes in $\mathbb{L}^2(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}[0, T], \mathbb{P} \otimes \lambda_L; U)$, where $U \subset \mathbb{R}^d$ and X follows the SDE

$$dX(s) = \mu_X(s, X(s), \nu(s))ds + \sigma_X(s, X(s), \nu(s))dW_s + \int_E \beta(s, X(s-), \nu(s), e)\lambda(ds, de).$$

To convert the control problem to its stochastic target counterpart, we need the following lemma, which is an adaptation of a result in [11].

Lemma 6.1 *Suppose Assumptions 2.1 and 2.2 hold. Define a stochastic target problem as follows:*

$$u(t, x) := \inf\{y \in \mathbb{R} : \exists(\nu, \alpha, \gamma) \in \mathcal{U}^t \times \mathcal{A}^t \times \Gamma^t \text{ s.t. } Y_{t,y}^{\alpha, \gamma}(T) \geq g(X_{t,x}^\nu(T))\}, \text{ where}$$

$$Y_{t,y}^{\alpha, \gamma}(\cdot) := y + \int_t^\cdot \alpha^\top(s) dW_s + \int_t^\cdot \int_E \gamma^\top(s, e) \tilde{\lambda}(ds, de)$$

and \mathcal{A}^t and Γ^t are the collections of \mathbb{R}^d -valued and $\mathbb{L}^2(E, \mathcal{E}, \hat{m}; \mathbb{R}^I)$ -valued processes, respectively, satisfying the admissibility conditions in Section 2. Then $u = \mathbf{u}$ on \mathbb{D} .

Proof Since \mathcal{A}^t and Γ^t satisfy the admissibility conditions, this stochastic target problem is well defined. In view of Lemma 2.1 in [11], it suffices to check that

$$\{g(X_{t,x}^\nu(T), \nu \in \mathcal{U}^t\} \subset \{M(T), M \in \mathcal{M}\}, \text{ where } \mathcal{M} := \{Y_{t,y}^{\alpha,\gamma}(\cdot) : y \in \mathbb{R}, \alpha \in \mathcal{A}^t, \gamma \in \Gamma^t\}. \quad (32)$$

In fact, by the martingale representation theorem, for any $\nu \in \mathcal{U}^t$, $\mathbb{E}[g(X_{t,x}^\nu(T))|\mathcal{F}^t]$ can be represented in the form of $Y_{t,y}^{\alpha,\gamma}$ for some $\alpha \in \mathcal{A}^t$ and $\gamma \in \Gamma_0^t$, where Γ_0^t is the collection of $\mathbb{L}^2(E, \mathcal{E}, \hat{m}; \mathbb{R}^I)$ -valued processes satisfying all of the admissibility conditions except for (2). In particular, $g(X_{t,x}^\nu(T)) = Y_{t,y}^{\alpha,\gamma}(T)$. Assume, contrary to (32), that there exists $\nu_0 \in \mathcal{U}^t$ such that

$$\mathbb{E}[g(X_{t,x}^{\nu_0}(T))|\mathcal{F}^t] = y + \int_t^\cdot \alpha_0^\top(s) dW_s + \int_t^\cdot \int_E \gamma_0^\top(s, e) \tilde{\lambda}(ds, de)$$

for some $y \in \mathbb{R}$, $\alpha_0 \in \mathcal{A}^t$ and $\gamma_0 \in \Gamma_0^t$, but (2) does not hold. In the equation above, $\mathbb{E}[g(X_{t,x}^{\nu_0}(T))|\mathcal{F}^t]$ can be chosen to be càdlàg, thanks to Theorem 1.3.13 in [14]. Then for $K > 2\|g\|_\infty$, there exists $\tau_0 \in \mathcal{T}_t$ such that $\mathbb{P}(|\int_E \gamma^\top(\tau_0, e) \lambda(\{\tau_0\}, de)| > K) > 0$. Suppose that $\mathbb{P}(\int_E \gamma^\top(\tau_0, e) \lambda(\{\tau_0\}, de) > K) > 0$.⁵ Let $M_0(\cdot) = \mathbb{E}[g(X_{t,x}^{\nu_0}(T))|\mathcal{F}^t]$. Therefore,

$$M_0(\tau_0) - M_0(\tau_0-) = \int_E \gamma^\top(\tau_0, e) \lambda(\{\tau_0\}, de) > K \text{ with positive probability.}$$

Since $|M_0|$ is bounded by $\|g\|_\infty < K/2$, we obtain a contradiction. \square

Let \mathbf{H}^* be the USC envelope of the LSC map $\mathbf{H} : \mathbb{D} \times \mathbb{R}^d \times \mathbb{M}^d \times C(\mathbb{D}) \rightarrow \mathbb{R}$ defined by

$$\mathbf{H} : (t, x, p, A, \varphi) \rightarrow \sup_{u \in U} \{-I[\varphi](t, x, u) - \mu_X^\top(t, x, u)p - \frac{1}{2} \text{Tr}[\sigma_X \sigma_X^\top(t, x, u)A]\}, \text{ where}$$

$$I[\varphi](t, x, u) = \sum_{1 \leq i \leq I} \int_E (\varphi(t, x + \beta_i(t, x, u, e)) - \varphi(t, x)) m_i(de).$$

⁵ If this does not hold, the integral is less than $-K$ with positive probability. Noticing this, we can carry out the proof in a similar manner when this assumption does not hold.

Theorem 6.1 *Under Assumptions 2.1 and 2.2, u^+ is a USC viscosity sub-solution of*

$$-\partial_t \varphi(t, x) + \mathbf{H} \varphi(t, x) \leq 0 \text{ on } \mathbb{D}_i$$

and $u^+(T-, x) \leq g(x)$ for all $x \in \mathbb{R}^d$. On the other hand, u^- is an LSC viscosity super-solution of

$$-\partial_t \varphi(t, x) + \mathbf{H}^* \varphi(t, x) \geq 0 \text{ on } \mathbb{D}_i$$

and $u^-(T-, \cdot)$ is an LSC viscosity super-solution of

$$(\varphi(x) - g(x)) \mathbb{1}_{\{\mathbf{H}^* \varphi(x) < \infty\}} \geq 0 \text{ on } \mathbb{R}^d.$$

Proof It is easy to check Assumption 2.4 for the stochastic target problem. Since g is bounded, we can check that all of the assumptions in the Appendix A are satisfied, which implies that Assumption 2.3 holds. From Theorem 3.1, u^+ is a USC viscosity sub-solution of $-\partial_t \varphi(t, x) + H_* \varphi(t, x) \leq 0$ on \mathbb{D}_i and u^- is an LSC viscosity super-solution of $-\partial_t \varphi(t, x) + H^* \varphi(t, x) \geq 0$ on \mathbb{D}_i . From Proposition 3.1 in [11], $H^* \leq \mathbf{H}^*$ and $H_* \geq \mathbf{H}$. This implies that the viscosity properties in the parabolic interior hold.

Also, by Theorem 4.1, $u^+(T-, \cdot)$ is a USC viscosity sub-solution of $\min\{\varphi(x) - g(x), \delta_* \varphi(x)\} \leq 0$ on \mathbb{R}^d and $u^-(T-, \cdot)$ is an LSC viscosity super-solution of $\min\{(\varphi(x) - g(x)) \mathbb{1}_{\{H^* \varphi(x) < \infty\}}, \delta^* \varphi(x)\} \geq 0$ on \mathbb{R}^d , where $\delta = \text{dist}(0, \mathbf{N}^c) - \text{dist}(0, \mathbf{N})$ and

$$\begin{aligned} \mathbf{N}(t, x, y, p, \varphi) &= \{(q, s) \in \mathbb{R}^d \times \mathbb{R} : \exists (u, a, r) \in U \times \mathbb{R}^d \times \mathbb{L}^2(E, \mathcal{E}, \hat{m}; \mathbb{R}^I) \text{ s.t. } q = a - \sigma_X^\top(t, x, u)p \\ &\text{and } s \leq \min_{1 \leq i \leq I} \{r_i(e) - \varphi(t, x + \beta_i(t, x, u, e)) + \varphi(t, x)\} \hat{m} - \text{a.s. } e \in E\}. \end{aligned}$$

Obviously, $\mathbf{N} = \mathbb{R}^d \times \mathbb{R}$. Therefore, $\delta = \infty$ and the boundary conditions hold. \square

The following two corollaries show that \mathbf{u} is the unique viscosity solution to its associated HJB equation.

We omit the proof, since it is the same as the proofs of Proposition 5.1 and Theorem 5.1.

Corollary 6.1 *Suppose that Assumptions 2.1 and 2.2 hold, $\mathbf{H} = \mathbf{H}^*$ on $\{\mathbf{H} < \infty\}$ and there exists an LSC function $\mathbf{G} : \mathbb{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d \times C(\mathbb{D}) \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} (a) \quad & \mathbf{H}(t, x, y, p, M, \varphi) < \infty \implies \mathbf{G}(t, x, y, p, M, \varphi) \leq 0, \\ (b) \quad & \mathbf{G}(t, x, y, p, M, \varphi) < 0 \implies \mathbf{H}(t, x, y, p, M, \varphi) < \infty. \end{aligned}$$

Then u^+ (resp. u^-) is a USC (resp. an LSC) viscosity sub-solution (resp. super-solution) of

$$\max\{-\partial_t \varphi(t, x) + \mathbf{H}\varphi(t, x), \mathbf{G}\varphi(t, x)\} = 0 \quad \text{on } \mathbb{D}_i$$

and $u^+(T-, \cdot)$ (resp. $u^-(T-, \cdot)$) is a USC (resp. an LSC) viscosity sub-solution (resp. super-solution) of

$$\max\{\varphi(x) - g(x), \mathbf{G}\varphi(x)\} = 0 \quad \text{on } \mathbb{R}^d.$$

Corollary 6.2 *Suppose that all of the assumptions in Corollary 6.1 hold. Additionally, assume that there is a comparison principle between USC sub-solutions and LSC super-solutions for the PDE*

$$\max\{\varphi(x) - g(x), \mathbf{G}\varphi(x)\} = 0 \quad \text{on } \mathbb{R}^d. \tag{33}$$

Then $u^+(T-, x) = u^-(T-, x) = \hat{\mathbf{g}}(x)$, where $\hat{\mathbf{g}}$ is the unique viscosity solution to (33). Furthermore, if the comparison principle holds for

$$\max\{-\partial_t \varphi(t, x) + \mathbf{H}\varphi(t, x), \mathbf{G}\varphi(t, x)\} = 0 \quad \text{on } \mathbb{D}_i, \tag{34}$$

then there exists a unique continuous viscosity solution \mathbf{V} to (34) with terminal condition $\mathbf{V}(T, x) = \hat{\mathbf{g}}(x)$ and $\mathbf{u}(t, x) = u(t, x) = u^+(t, x) = u^-(t, x) = \mathbf{V}(t, x)$ for $(t, x) \in \mathbb{D}_i$.

7 Conclusions

In this paper, stochastic target problems in a jump diffusion setup are analyzed by using stochastic Perron's method, which had been recently developed to analyze the classical stochastic control problems. In fact, we

using the fact that ordinary stochastic control problems can be embedded into stochastic target problems we extended that analysis to cover to processes in which both the diffusions and jumps are controlled. Our future research will focus on extending the analysis to stochastic target games. In the formulation of such problems, a strategic player tries to find a strategy such that the controlled process reaches a given target no matter what the opponent's control is. Of particular importance is the set-up in which one of the players is a stopper, whose aim is to get to the target at a stopping time instead of a fixed horizon.

Appendix A

We provide sufficient conditions for the nonemptiness of \mathbb{U}^+ and \mathbb{U}^- .

Assumption A.1 g is bounded.

Assumption A.2 There exists $u_0 \in U$ such that $\sigma_Y(t, x, y, u_0) = 0$ and $b(t, x, y, u_0(e), e) = 0$ for all $(t, x, y, e) \in \mathbb{D} \times \mathbb{R} \times E$.

Remark A.1 In the context of super-hedging in mathematical finance, the assumption above is equivalent to restricting trading to the riskless assets.

Proposition A.1 Under Assumptions 2.1, 2.2, A.1 and A.2, \mathbb{U}^+ is not empty.

Proof **Step 1.** In this step we assume that μ_Y is non-decreasing in its y -variable. We will show that $w(t, x) = \gamma - e^{kt}$ is a stochastic super-solution for some choice of k and γ .

By the linear growth condition on μ_Y in Assumption 2.2, there exists $L > 0$ such that $|\mu_Y(t, x, y, u_0)| \leq L(1 + |y|)$, where u_0 is the element in U in Assumption A.2. Choose $k \geq 2L$ and γ such that $-e^{kT} + \gamma \geq \|g\|_\infty$. Then $w(T, x) \geq g(x)$. It suffices to show that for any $(t, x, y) \in \mathbb{D} \times \mathbb{R}$, $\tau \in \mathcal{T}_t$, $\nu \in \mathcal{U}^t$ and $\rho \in \mathcal{T}_\tau$,

$$Y(\rho) \geq w(\rho, X(\rho)) \quad \mathbb{P}\text{-a.s. on } \{Y(\tau) \geq w(\tau, X(\tau))\}, \text{ where } X := X_{t,x}^{\nu \otimes \tau u_0}, Y := Y_{t,x,y}^{\nu \otimes \tau u_0}. \quad (35)$$

Let $A = \{Y(\tau) > w(\tau, X(\tau))\}$, $V(s) = w(s, X(s))$ and $\Gamma(s) = (V(s) - Y(s)) \mathbb{1}_A$. Therefore, for $s \geq \tau$,

$$\begin{aligned} dY(s) &= \mu_Y(s, X(s), Y(s), u_0) ds, \quad dV(s) = -ke^{ks} ds, \quad \Gamma(s) = \mathbb{1}_A \int_\tau^s (\xi(q) + \Delta(q)) dq, \text{ where} \\ \Delta(s) &:= -ke^{ks} - \mu_Y(s, X(s), Y(s), u_0) \leq -ke^{ks} - \mu_Y(s, X(s), -e^{ks}, u_0) \leq -ke^{ks} + L(1 + e^{ks}) \leq 0, \\ \xi(s) &:= \mu_Y(s, X(s), V(s), u_0) - \mu_Y(s, X(s), Y(s), u_0). \end{aligned} \quad (36)$$

Therefore, from (36) it holds that

$$\Gamma(s) \leq \mathbb{1}_A \int_\tau^s \xi(q) dq \quad \text{and} \quad \Gamma^+(s) \leq \mathbb{1}_A \int_\tau^s \xi^+(q) dq \quad \text{for } s \geq \tau.$$

From the Lipschitz continuity of μ_Y in y -variable in Assumption 2.2,

$$\Gamma^+(s) \leq \mathbb{1}_A \int_\tau^s \xi^+(q) dq \leq \int_\tau^s L_0 \Gamma^+(q) dq \quad \text{for } s \geq \tau,$$

where L_0 is the Lipschitz constant of μ_Y with respect to y . Note that we use the assumption that μ_Y is non-decreasing in its y -variable to obtain the second inequality. Since $\Gamma^+(\tau) = 0$, an application of Grönwall's Inequality implies that $\Gamma^+(\rho) \leq 0$, which further implies that (35) holds.

Step 2. We get rid of our assumption on μ_Y from Step 1 by following a proof similar to those in [12] and [15]. For $c > 0$, define $\tilde{Y}_{t,x,y}^\nu$ as the strong solution of

$$d\tilde{Y}(s) = \tilde{\mu}_Y(s, X_{t,x}^\nu(s), \tilde{Y}(s), \nu(s)) ds + \tilde{\sigma}_Y^\top(s, X_{t,x}^\nu(s), \tilde{Y}(s), \nu(s)) dW_s + \int_E \tilde{b}^\top(s, X_{t,x}^\nu(s-), \tilde{Y}(s-), \nu_1(s), \nu_2(s, e), e) \lambda(ds, de)$$

with initial data $\tilde{Y}(t) = y$, where

$$\tilde{\mu}_Y(t, x, y, u) := cy + e^{ct} \mu_Y(t, x, e^{-ct} y, u), \quad \tilde{\sigma}_Y(t, x, y, u) := e^{ct} \sigma_Y(t, x, e^{-ct} y, u), \quad \tilde{b}(t, x, y, u(e), e) := e^{ct} b(t, x, e^{-ct} y, u(e), e).$$

Therefore,

$$\tilde{Y}_{t,x,y}^\nu(s)e^{-cs} = Y_{t,x,ye^{-ct}}^\nu(s), \quad t \leq s \leq T.$$

Let $\tilde{u}(t, x) = \inf\{y \in \mathbb{R} : \exists \nu \in \mathcal{U}^t, \text{ s.t. } \tilde{Y}_{t,x,y}^\nu(T) \geq \tilde{g}(X_{t,x}^\nu(T)) \text{ -a.s.}\}$, where $\tilde{g}(x) = e^{cT}g(x)$. Therefore, $\tilde{u}(t, x) = e^{ct}u(t, x)$. Since μ_Y is Lipschitz in y , we can choose $c > 0$ so that

$$\tilde{\mu}_Y : (t, x, y, u) \mapsto cy + e^{ct}\mu_Y(t, x, e^{-ct}y, u)$$

is non-decreasing in y . Moreover, all the properties of $\tilde{\mu}_Y, \tilde{\sigma}_Y$ and \tilde{b} in Assumption 2.2 still hold. We replace μ_Y, σ_Y and b in all of the equations and definitions in Section 2 with $\tilde{\mu}_Y, \tilde{\sigma}_Y$ and \tilde{b} , we get \tilde{H}^* and \tilde{H}_* . Let $\tilde{\mathbb{U}}^+$ be the set of stochastic super-solutions of

$$-\partial_t \varphi(t, x) + \tilde{H}^* \varphi(t, x) \geq 0 \quad \text{on } \mathbb{D}_i.$$

It is easy to see that $w \in \mathbb{U}^+$ if and only if $\tilde{w}(t, x) := e^{ct}w(t, x) \in \tilde{\mathbb{U}}^+$. From Step 1, $\tilde{\mathbb{U}}^+$ is not empty. Thus, \mathbb{U}^+ is not empty. \square

Assumption A.3 *There is $C \in \mathbb{R}$ such that for all $(t, x, y, u, e) \in \mathbb{D} \times \mathbb{R} \times U \times E$,*

$$\left| \mu_Y(t, x, y, u) + \int_E b^\top(t, x, y, u(e), e)m(de) \right| \leq C(1 + |y|).$$

Proposition A.2 *Under Assumptions 2.1, 2.2, A.1 and A.3, \mathbb{U}^- is not empty.*

Proof Assume that

$$\mu_Y(t, x, y, u) + \int_E b^\top(t, x, y, u(e), e)m(de)$$

is non-decreasing in its y -variable. We could remove this assumption by using the argument from previous proposition.

Choose $k \geq 2C$ (C is the constant in Assumption A.3) and $\gamma > 0$ such that $e^{kT} - \gamma < -\|g\|_\infty$. Let $w(t, x) = e^{kx} - \gamma$. Notice that w is continuous, has polynomial growth in x and $w(T, \cdot) \leq g(\cdot)$. It suffices to show that for any $(t, x, y) \in \mathbb{D} \times \mathbb{R}$, $\tau \in \mathcal{T}_t$ and $\nu \in \mathcal{U}^t$, it holds that $\mathbb{P}(Y(\rho) < w(\rho, X(\rho))|B) > 0$ for all $\rho \in \mathcal{T}_\tau$ and $B \subset \{Y(\tau) < w(\tau, X(\tau))\}$ satisfying $B \in \mathcal{F}_\tau^t$ and $\mathbb{P}(B) > 0$, where $X := X_{t,x}^\nu$ and $Y := Y_{t,x,y}^\nu$. Define

$$M(\cdot) = Y(\cdot) - \int_\tau^\cdot K(s)ds, \quad V(s) = w(s, X(s)), \quad A = \{Y(\tau) < w(\tau, X(\tau))\}, \quad \Gamma(s) = (Y(s) - V(s)) \mathbb{1}_A, \quad \text{where}$$

$$K(s) := \mu_Y(s, X(s), Y(s), \nu(s)) + \int_E b^\top(s, X(s-), Y(s-), \nu_1(s), \nu_2(s, e), e)m(de),$$

$$\tilde{K}(s) := \mu_Y(s, X(s), V(s), \nu(s)) + \int_E b^\top(s, X(s-), V(s-), \nu_1(s), \nu_2(s, e), e)m(de).$$

It is easy to see that M is a martingale after τ . Due to the facts that $A \in \mathcal{F}_\tau^t$ and $dV(s) = ke^{ks}ds$, we further know

$$\mathbb{1}_A \left(Y(\cdot) - V(\cdot) + \int_\tau^\cdot ke^{ks} - K(s)ds \right) \quad \text{is a super-martingale after } \tau. \quad (37)$$

Since Assumption A.3 holds and $\mu_Y(t, x, y, u) + \int_E b^\top(t, x, y, u(e), e)m(de)$ is non-decreasing in y ,

$$\tilde{K}(s) \leq \mu_Y(s, X(s), e^{ks}, \nu(s)) + \int_E b^\top(s, X(s-), e^{ks}, \nu_1(s), \nu_2(s, e), e)m(de) \leq 2Ce^{ks}.$$

Therefore, it follows from (37) and the inequality above that

$$\widetilde{M}(\cdot) := \mathbb{1}_A \left(Y(\cdot) - V(\cdot) - \int_\tau^\cdot \xi(s)ds \right) \quad \text{is a super-martingale after } \tau, \quad \text{where } \xi(s) := K(s) - \tilde{K}(s). \quad (38)$$

Since $\widetilde{M}(\tau) < 0$ on B , there exists a non-null set $F \subset B$ such that $\widetilde{M}(\rho) < 0$ on F . By the definition of \widetilde{M} in (38), we get

$$\Gamma(\rho) < \mathbb{1}_A \int_\tau^\rho \xi(s)ds \quad \text{on } F. \quad (39)$$

Therefore,

$$\Gamma^+(\rho) \leq \mathbb{1}_A \int_\tau^\rho \xi^+(s)ds \leq \int_\tau^\rho L_0 \Gamma^+(s)ds \quad \text{on } F. \quad (40)$$

By Grönwall's Inequality, $\Gamma^+(\tau) = 0$ implies that $\Gamma^+(\rho) = 0$ on F . More precisely, for $\omega \in F$ (\mathbb{P} -a.s.), $\Gamma^+(s)(\omega) = 0$ for $s \in [\tau(\omega), \rho(\omega)]$. This implies that we can replace the inequalities with equalities in (40). Therefore, by (39), $\Gamma(\rho) < 0$ on F , which yields $\mathbb{P}(Y(\rho) < w(\rho, X(\rho))|B) > 0$. \square

Appendix B

Proof of Theorem 3.1

Step 1 (u^+ is a viscosity sub-solution). Assume, on the contrary, that for some $(t_0, x_0) \in \mathbb{D}_i$ and $\varphi \in C^{1,2}(\mathbb{D})$ satisfying $0 = (u^+ - \varphi)(t_0, x_0) = \max_{\mathbb{D}_i}(u^+ - \varphi)$, we have

$$4\eta := -\partial_t \varphi(t_0, x_0) + H_* \varphi(t_0, x_0) > 0. \quad (41)$$

From Lemma 3.2, there exists a non-increasing sequence $\mathbb{U}^+ \ni w_k \searrow u^+$. Fix such a sequence $\{w_k\}_{k=1}^\infty$ and an arbitrary stochastic sub-solution w_- . Let $\tilde{\varphi}(t, x) = \varphi(t, x) + \iota|x - x_0|^{n_0}$.⁶ We can choose $n_0 \geq 2$ such that for any $\iota > 0$,

$$\min_{0 \leq t \leq T} (\tilde{\varphi}(t, x) - w_1(t, x)) \rightarrow \infty \text{ as } |x| \rightarrow \infty. \quad (42)$$

We can do this because $\varphi(t, x)$ is bounded from below by w_- (which has polynomial growth in x) and w_1 has polynomial growth in x . Since $(\mathcal{N}_{\varepsilon, \eta})_{\varepsilon \geq 0}$ is non-decreasing in ε , we know

$$H_*(\Theta, \varphi) = \liminf_{\substack{\Theta' \rightarrow \Theta, \psi \xrightarrow{\text{usc}} \varphi \\ \eta \searrow 0}} H_{0, \eta}(\Theta', \psi).$$

By (4) and (41), we can find $\varepsilon > 0$, $\eta > 0$ and $\iota > 0$ such that for all (t, x, y) satisfying $(t, x) \in B_\varepsilon(t_0, x_0)$ and $|y - \tilde{\varphi}(t, x)| \leq \varepsilon$, $\mu_Y(t, x, y, u) - \mathcal{L}^u \tilde{\varphi}(t, x) \geq 2\eta$ for some $u \in \mathcal{N}_{0, \eta}(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi})$. Fix ι . Note that (t_0, x_0) is still a strict maximizer of $u^+ - \tilde{\varphi}$ over \mathbb{D}_i . For ε sufficiently small, Assumption 2.4 implies that there exists a locally Lipschitz map $\hat{\nu}$ such that

$$\hat{\nu}(t, x, y, D\tilde{\varphi}(t, x)) \in \mathcal{N}_{0, \eta}(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi}) \text{ and} \quad (43)$$

$$\begin{aligned} \mu_Y(t, x, y, \hat{\nu}(t, x, y, D\tilde{\varphi}(t, x))) - \mathcal{L}^{\hat{\nu}(t, x, y, D\tilde{\varphi}(t, x))} \tilde{\varphi}(t, x) &\geq \eta \\ \text{for all } (t, x, y) \in \mathbb{D}_i \times \mathbb{R} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \tilde{\varphi}(t, x)| \leq \varepsilon. \end{aligned} \quad (44)$$

In the arguments above, choose ε small enough such that $\text{cl}(B_\varepsilon(t_0, x_0)) \cap \mathbb{D}_T = \emptyset$. Since (42) holds, there exists $R_0 > \varepsilon$ such that $\tilde{\varphi} > w_1 + \varepsilon \geq w_k + \varepsilon$ on $\mathbb{O} := \mathbb{D} \setminus [0, T] \times \text{cl}(B_{R_0}(x_0))$ for all k . On the compact set $\mathbb{T} := [0, T] \times \text{cl}(B_{R_0}(x_0)) \setminus B_{\varepsilon/2}(t_0, x_0)$, we know that $\tilde{\varphi} > u^+$ and the minimum of $\tilde{\varphi} - u^+$ is attained since u^+ is USC. Therefore, $\tilde{\varphi} > u^+ + 2\alpha$ on \mathbb{T} for some $\alpha > 0$. By a Dini-type argument, for large enough n , we have $\tilde{\varphi} > w_n + \alpha$ on \mathbb{T} and $\tilde{\varphi} > w_n - \varepsilon$ on $\text{cl}(B_{\varepsilon/2}(t_0, x_0))$. For simplicity, fix such an n and set $w = w_n$. In short,

$$\tilde{\varphi} > w + \varepsilon \text{ on } \mathbb{O}, \tilde{\varphi} > w + \alpha \text{ on } \mathbb{T} \text{ and } \tilde{\varphi} > w - \varepsilon \text{ on } \text{cl}(B_{\varepsilon/2}(t_0, x_0)). \quad (45)$$

For $\kappa \in]0, \varepsilon \wedge \alpha[$, define

$$w^\kappa := \begin{cases} (\tilde{\varphi} - \kappa) \wedge w & \text{on } \text{cl}(B_\varepsilon(t_0, x_0)), \\ w & \text{outside } \text{cl}(B_\varepsilon(t_0, x_0)). \end{cases}$$

Observing that $w^\kappa(t_0, x_0) = \tilde{\varphi}(t_0, x_0) - \kappa < u^+(t_0, x_0)$, we could obtain a contradiction if we could show that $w^\kappa \in \mathbb{U}^+$. Obviously, w^κ is continuous, has polynomial growth in x and $w^\kappa(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$. Fix $(t, x, y) \in \mathbb{D}_i \times \mathbb{R}$, $\nu \in \mathcal{U}^t$ and $\tau \in \mathcal{T}_t$.⁷ Now our goal is to construct an admissible control $\tilde{\nu}$ such that w^κ and the processes (X, Y) controlled by $\nu \otimes_\tau \tilde{\nu}$ satisfy the property in the definition of stochastic super-solutions.

Let $A = \{w^\kappa(\tau, X_{t,x}^\nu(\tau)) = w(\tau, X_{t,x}^\nu(\tau))\}$. On A , let $\tilde{\nu}$ be $\tilde{\nu}_1$, which is “optimal” for w starting at τ . We get the existence of $\tilde{\nu}_1$ since $w \in \mathbb{U}^+$. On A^c , by an argument similar to that in [12] (see Step 1.1 of Theorem 3.1’s proof), we can construct an admissible control $\nu_0 \in \mathcal{U}^t$ such that

$$\begin{aligned} \nu_0(s) &:= \hat{\nu}\left(s, X_{t,x}^{\nu \otimes_\tau \nu_0}(s), Y_{t,x,y}^{\nu \otimes_\tau \nu_0}(s), D\tilde{\varphi}(s, X_{t,x}^{\nu \otimes_\tau \nu_0}(s))\right) \text{ for } \tau \leq s < \theta,^8 \text{ where } \theta = \theta_1 \wedge \theta_2 \text{ and} \\ \theta_1 &:= \inf \left\{ s \in [\tau, T] : (s, X_{t,x}^{\nu \otimes_\tau \nu_0}(s)) \notin B_{\varepsilon/2}(t_0, x_0) \right\} \wedge T, \quad \theta_2 := \inf \left\{ s \in [\tau, T] : \left| Y_{t,x,y}^{\nu \otimes_\tau \nu_0}(s) - \tilde{\varphi}(s, X_{t,x}^{\nu \otimes_\tau \nu_0}(s)) \right| \geq \varepsilon \right\} \wedge T. \end{aligned}$$

In the construction of ν_0 , we take advantage of Assumption 2.2 and the Lipschitz continuity of $\hat{\nu}$ which guarantee the existence of $X_{t,x}^{\nu \otimes_\tau \nu_0}$ and $Y_{t,x,y}^{\nu \otimes_\tau \nu_0}$. Since $X_{t,x}^{\nu \otimes_\tau \nu_0}$ and $Y_{t,x,y}^{\nu \otimes_\tau \nu_0}$ are càdlàg, it is easy to check that $\theta \in \mathcal{T}_\tau$. We also see that

$$(\theta_1, X_{t,x}^{\nu \otimes_\tau \nu_0}(\theta_1)) \notin B_{\varepsilon/2}(t_0, x_0), \quad \left| Y_{t,x,y}^{\nu \otimes_\tau \nu_0}(\theta_2) - \tilde{\varphi}(\theta_2, X_{t,x}^{\nu \otimes_\tau \nu_0}(\theta_2)) \right| \geq \varepsilon, \quad (46)$$

$$(\theta_1, X_{t,x}^{\nu \otimes_\tau \nu_0}(\theta_1-)) \in \text{cl}(B_{\varepsilon/2}(t_0, x_0)), \quad \left| Y_{t,x,y}^{\nu \otimes_\tau \nu_0}(\theta_2-) - \tilde{\varphi}(\theta_2, X_{t,x}^{\nu \otimes_\tau \nu_0}(\theta_2-)) \right| \leq \varepsilon. \quad (47)$$

Let $\tilde{\nu}^\theta$ be the “optimal” control for w starting at θ . We define $\tilde{\nu}$ on A^c by $\nu_0 \otimes_\theta \tilde{\nu}^\theta$. In short,

$$\tilde{\nu} := \left(\mathbb{1}_A \tilde{\nu}_1 + \mathbb{1}_{A^c} (\nu_0 \mathbb{1}_{[\tau, \theta]} + \mathbb{1}_{[\theta, T]} \tilde{\nu}^\theta) \right) \mathbb{1}_{[\tau, T]}.$$

⁶ Since we will fix n_0 and ι later, we still use the notation $\tilde{\varphi}$ when without ambiguity despite the fact that the function depends on n_0 and ι .

⁷ Here we choose $(t, x) \in \mathbb{D}_i$ since the case $(t, x) \in \mathbb{D}_T$ is trivial.

It is not difficult to check that $\tilde{\nu} \in \mathcal{U}^t$. To prove that the above construction works, we next show that $Y(\rho) \geq w^\kappa(\rho, X(\rho))$ \mathbb{P} -a.s. on $\{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$, where $X := X_{t,x}^{\nu \otimes \tau \tilde{\nu}}$ and $Y := Y_{t,x,y}^{\nu \otimes \tau \tilde{\nu}}$. Corresponding to the construction of $\tilde{\nu}$ on A and A^c , we consider the following two cases:

(i) **On the set** $A \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$. We have $Y(\tau) \geq w(\tau, X(\tau))$. From the definition of ν on A and the fact that $w \in \mathbb{U}^+$, we know

$$Y(\rho) = Y_{t,x,y}^{\nu \otimes \tau \tilde{\nu} 1}(\rho) \geq w(\rho, X_{t,x}^{\nu \otimes \tau \tilde{\nu} 1}(\rho)) \geq w^\kappa(\rho, X(\rho)) \quad \mathbb{P}\text{-a.s. on } A \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}.$$

(ii) **On the set** $A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$. Letting $\Gamma(s) := Y(s) - \tilde{\varphi}(s, X(s))$, we use Itô's formula and the definition of ν_0 to obtain

$$\Gamma(\cdot \wedge \theta) = \Gamma(\tau) + \int_\tau^{\cdot \wedge \theta} \int_E \bar{J}^{\nu_0(s), e}(s, Z(s-), \tilde{\varphi})^\top \lambda(ds, de) + \int_\tau^{\cdot \wedge \theta} \left(\mu_Y(s, Z(s), \nu_0(s)) - \mathcal{L}^{\nu_0(s)} \tilde{\varphi}(s, X(s)) \right) ds$$

on $A \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$. Therefore, by (43), (44), (47) and the definition of θ , we know that $\Gamma(\cdot \wedge \theta)$ is non-decreasing on $[\tau, T]$. This implies that

$$Y(\theta) - \tilde{\varphi}(\theta, X(\theta)) + \kappa \geq Y(\tau) - \tilde{\varphi}(\tau, X(\tau)) + \kappa \geq 0 \quad \text{on } A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}. \quad (48)$$

Since $(\theta_1, X(\theta_1)) \notin B_{\varepsilon/2}(t_0, x_0)$, we know

$$0 \leq Y(\theta_1) - \tilde{\varphi}(\theta_1, X(\theta_1)) + \kappa \leq Y(\theta_1) - w(\theta_1, X(\theta_1)) \quad \text{on } \{\theta_1 \leq \theta_2\} \cap A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\} \quad (49)$$

from (45). On the other hand, it holds that $Y(\theta_2) - \tilde{\varphi}(\theta_2, X(\theta_2)) \geq \varepsilon$ on $\{\theta_1 > \theta_2\} \cap A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$ due to (46) and (48). Therefore, since $\tilde{\varphi} > w - \varepsilon$ on $\text{cl}(B_{\varepsilon/2}(t_0, x_0))$ and (47) holds,

$$Y(\theta_2) - w(\theta_2, X(\theta_2)) \geq \varepsilon + \tilde{\varphi}(\theta_2, X(\theta_2)) - w(\theta_2, X(\theta_2)) > 0 \quad \text{on } \{\theta_1 > \theta_2\} \cap A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}. \quad (50)$$

Combining (49) and (50), we obtain $Y(\theta) - w(\theta, X(\theta)) \geq 0$ on $A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$. Therefore, from the definition of $\tilde{\nu}^\theta$,

$$Y(\rho \vee \theta) - w^\kappa(\rho \vee \theta, X(\rho \vee \theta)) \geq Y(\rho \vee \theta) - w(\rho \vee \theta, X(\rho \vee \theta)) \geq 0 \quad \text{on } A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}. \quad (51)$$

Also, the monotonicity of $\Gamma(\cdot \wedge \theta)$ implies that $Y(\rho \wedge \theta) - \tilde{\varphi}(\rho \wedge \theta, X(\rho \wedge \theta)) + \kappa \geq 0$ on $A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$. This means that

$$\mathbb{1}_{\{\rho < \theta\}} (Y(\rho) - w^\kappa(\rho, X(\rho))) \geq 0 \quad \text{on } A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}. \quad (52)$$

From (51) and (52), we get $Y(\rho) - w^\kappa(\rho, X(\rho)) \geq 0$ on $A^c \cap \{Y(\tau) \geq w^\kappa(\tau, X(\tau))\}$.

Step 2 (u^- is a viscosity super-solution). Let $(t_0, x_0) \in \mathbb{D}_i$ satisfy $0 = (u^- - \varphi)(t_0, x_0) = \min_{\mathbb{D}_i} (u^- - \varphi)$ for some $\varphi \in C^{1,2}(\mathbb{D})$. For the sake of contradiction, assume that

$$-2\eta := -\partial_t \varphi(t_0, x_0) + H^* \varphi(t_0, x_0) < 0. \quad (53)$$

Let $\{w_k\}_{k=1}^\infty$ be a sequence in \mathbb{U}^- such that $w_k \nearrow u^-$. Let $\tilde{\varphi}(t, x) := \varphi(t, x) - \iota|x - x_0|^{n_0}$, where we choose $n_0 \geq 2$ such that for all $\iota > 0$,

$$\max_{0 \leq t \leq T} (\tilde{\varphi}(t, x) - w_1(t, x)) \rightarrow -\infty \quad \text{and} \quad \max_{0 \leq t \leq T} \tilde{\varphi}(t, x) \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty.^9 \quad (54)$$

By (53), the upper semi-continuity of H^* and the fact that $\tilde{\varphi} \xrightarrow{\text{u.c.}} \varphi$ as $\iota \rightarrow 0$, we can find $\varepsilon > 0$, $\eta > 0$ and $\iota > 0$ such that

$$\begin{aligned} \mu_Y(t, x, y, u) - \mathcal{L}^u \tilde{\varphi}(t, x) &\leq -\eta \text{ for all } u \in \mathcal{N}_{\varepsilon, -\eta}(t, x, y, D\tilde{\varphi}(t, x), \tilde{\varphi}) \\ \text{and } (t, x, y) &\in \mathbb{D}_i \times \mathbb{R} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \tilde{\varphi}(t, x)| \leq \varepsilon. \end{aligned} \quad (55)$$

Fix ι . Note that (t_0, x_0) is still a strict minimizer of $u^- - \tilde{\varphi}$. Since (54) holds, there exists $R_0 > \varepsilon$ such that

$$\tilde{\varphi} < w_1 - \varepsilon \leq w_k - \varepsilon \quad \text{on } \mathbb{O} := \mathbb{D} \setminus [0, T] \times \text{cl}(B_{R_0}(x_0)).$$

On the compact set $\mathbb{T} := [0, T] \times \text{cl}(B_{R_0}(x_0)) \setminus B_{\varepsilon/2}(t_0, x_0)$, we know that $\tilde{\varphi} < u^-$ and the maximum of $\tilde{\varphi} - u^-$ is attained since u^- is LSC. Therefore, $\tilde{\varphi} < u^- - 2\alpha$ on \mathbb{T} for some $\alpha > 0$. By a Dini-type argument, for large enough n , we have $\tilde{\varphi} < w_n - \alpha$ on \mathbb{T} and $\tilde{\varphi} < w_n + \varepsilon$ on $\text{cl}(B_{\varepsilon/2}(t_0, x_0))$. For simplicity, fix such an n and set $w = w_n$. In short,

$$\tilde{\varphi} < w - \varepsilon \quad \text{on } \mathbb{O}, \quad \tilde{\varphi} < w - \alpha \quad \text{on } \mathbb{T} \quad \text{and} \quad \tilde{\varphi} < w + \varepsilon \quad \text{on } \text{cl}(B_{\varepsilon/2}(t_0, x_0)). \quad (56)$$

For $\kappa \in]0, \alpha \wedge \varepsilon[$, define

$$w^\kappa := \begin{cases} (\tilde{\varphi} + \kappa) \vee w & \text{on } \text{cl}(B_\varepsilon(t_0, x_0)), \\ w & \text{outside } \text{cl}(B_\varepsilon(t_0, x_0)). \end{cases}$$

Noticing that $w^\kappa(t_0, x_0) \geq \tilde{\varphi}(t_0, x_0) + \kappa > u^-(t_0, x_0)$, we will obtain a contradiction if we show that $w^\kappa \in \mathbb{U}^-$. Obviously, w^κ is continuous, has polynomial growth in x and $w^\kappa(T, x) \leq g(x)$ for all $x \in \mathbb{R}^d$. Fix $(t, x, y) \in \mathbb{D}_i \times \mathbb{R}$, $\nu \in \mathcal{U}^t$ and $\tau \in \mathcal{T}_t$. Our

⁹ The existence of n_0 follows as in Step1.

goal is to show that

$$\mathbb{P}(Y(\rho) < w^\kappa(\rho, X(\rho)) | B) > 0$$

for all $\rho \in \mathcal{T}_\tau$ and $B \subset \{Y(\tau) < w^\kappa(\tau, X(\tau))\}$ satisfying $B \in \mathcal{F}_\tau^t$ and $\mathbb{P}(B) > 0$, where $X := X_{t,x}^\nu$ and $Y := Y_{t,x,y}^\nu$. Let $A = \{w^\kappa(\tau, X(\tau)) = w(\tau, X(\tau))\}$ and set

$$\begin{aligned} E &= \{Y(\tau) < w^\kappa(\tau, X(\tau))\}, \quad E_0 = E \cap A, \quad E_1 = E \cap A^c, \\ G &= \{Y(\rho) < w^\kappa(\rho, X(\rho))\}, \quad G_0 = \{Y(\rho) < w(\rho, X(\rho))\}. \end{aligned}$$

Then $E = E_0 \cup E_1$, $E_0 \cap E_1 = \emptyset$ and $G_0 \subset G$. To prove that $w^\kappa \in \mathbb{U}^-$, it suffices to show that $\mathbb{P}(G \cap B) > 0$. As in [12] and [3], we will show $\mathbb{P}(B \cap E_0) > 0 \implies \mathbb{P}(G \cap B \cap E_0) > 0$ and $\mathbb{P}(B \cap E_1) > 0 \implies \mathbb{P}(G \cap B \cap E_1) > 0$. This, together with the facts $\mathbb{P}(B) = \mathbb{P}(B \cap E_0) + \mathbb{P}(B \cap E_1) > 0$ and $\mathbb{P}(G \cap B) = \mathbb{P}(G \cap B \cap E_0) + \mathbb{P}(G \cap B \cap E_1)$, implies that $\mathbb{P}(G \cap B) > 0$.

(i) Assume that $\mathbb{P}(B \cap E_0) > 0$. Since $B \cap E_0 \subset \{Y(\tau) < w(\tau, X(\tau))\}$ and $B \cap E_0 \in \mathcal{F}_\tau^t$, $\mathbb{P}(G_0 | B \cap E_0) > 0$ from the definition of \mathbb{U}^- . This further implies that $\mathbb{P}(G \cap B \cap E_0) \geq \mathbb{P}(G_0 \cap B \cap E_0) > 0$.

(ii) Assume that $\mathbb{P}(B \cap E_1) > 0$. Let $\theta = \theta_1 \wedge \theta_2$, where

$$\theta_1 := \inf \{s \in [\tau, T] : (s, X(s)) \notin B_{\varepsilon/2}(t_0, x_0)\} \wedge T, \quad \theta_2 := \inf \{s \in [\tau, T] : |Y(s) - \tilde{\varphi}(s, X(s))| \geq \varepsilon\} \wedge T.$$

Since X and Y are càdlàg processes, we know that $\theta \in \mathcal{T}_\tau$. The following also hold:

$$(\theta_1, X(\theta_1)) \notin B_{\varepsilon/2}(t_0, x_0), \quad |Y(\theta_2) - \tilde{\varphi}(\theta_2, X(\theta_2))| \geq \varepsilon, \quad (57)$$

$$(\theta_1, X(\theta_1-)) \in \text{cl}(B_{\varepsilon/2}(t_0, x_0)), \quad |Y(\theta_2-) - \tilde{\varphi}(\theta_2, X(\theta_2-))| \leq \varepsilon. \quad (58)$$

Let

$$\begin{aligned} c_i^e(s) &= J_i^{u,e}(s, X(s-), Y(s-), \tilde{\varphi}), \quad d_i(s) = \int_E c_i^e(s) m_i(de), \quad d(s) = \sum_{i=1}^I d_i(s), \\ a(s) &= \mu_Y(s, X(s), Y(s), \nu(s)) - \mathcal{L}^{\nu(s)} \tilde{\varphi}(s, X(s)), \quad \pi(s) = N^{\nu(s)}(s, X(s), Y(s), D\tilde{\varphi}(s, X(s))), \\ A_0 &= \{s \in [\tau, \theta] : |\pi(s)| \leq \varepsilon\}, \quad A_{3,i} = \{(s, e) \in [\tau, \theta] \times E : c_i^e(s) \leq -\eta/2\}, \\ A_1 &= \{s \in [\tau, \theta] : c_i^e(s) \geq -\eta \text{ for } \hat{m} - a.s. \ e \in E \text{ for all } i = 1, \dots, I\}, \quad A_2 = (A_1)^c. \end{aligned}$$

We then set

$$L(\cdot) := \mathcal{E} \left(\int_t^{\cdot \wedge \theta} \int_E \sum \delta_i^e(s) \tilde{\lambda}_i(ds, de) + \int_t^{\cdot \wedge \theta} \alpha^\top(s) dW_s \right),$$

where $\mathcal{E}(\cdot)$ denotes the Doléans-Dade exponential and

$$\begin{aligned} x^+ &:= \max\{0, x\}, \quad x^- := \max\{0, -x\}, \quad \alpha(s) := -\frac{a(s) + d(s)}{|\pi(s)|^2} \pi(s) \mathbb{1}_{A_0^c}(s), \quad M_i(s) := \int_E \mathbb{1}_{A_{3,i}}(s, e) m_i(de), \\ K_i(s, e) &:= \begin{cases} \frac{\mathbb{1}_{A_{3,i}}(s, e)}{M_i(s)} & \text{if } M_i(s) > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \delta_i^e(s) := \left(\frac{\eta}{2(|d(s)| + 1)} - 1 + \mathbb{1}_{A_2}(s) \cdot \frac{2a(s)^+ + \eta}{\eta} \cdot K_i(s, e) \right) \mathbb{1}_{A_0}(s). \end{aligned}$$

If $s \in A_2$, then it follows from Assumption 2.1 and definitions of A_2 and $A_{3,i}$ that

$$M_{i_0}(s) > 0 \text{ for some } i_0 \in \{1, 2, \dots, I\}. \quad (59)$$

Obviously, L is a nonnegative local martingale on $[t, T]$. Therefore, it is a super-martingale. Let $\Gamma(s) := Y(s) - \tilde{\varphi}(s, X(s)) - \kappa$. Applying Itô's formula, we get

$$\begin{aligned} \Gamma(\cdot \wedge \theta) L(\cdot \wedge \theta) &= \Gamma(\tau) L(\tau) + \int_\tau^{\cdot \wedge \theta} L(s) \left\{ (a(s) + d(s)) \mathbb{1}_{A_0}(s) + \int_E \sum c_i^e(s) \delta_i^e(s) m_i(de) \right\} ds \\ &\quad + \int_\tau^{\cdot \wedge \theta} \int_E \sum L(s) \{ c_i^e(s) + \Gamma(s) \delta_i^e(s) + c_i^e(s) \delta_i^e(s) \} \tilde{\lambda}(ds, de) + \int_\tau^{\cdot \wedge \theta} L(s) (\pi(s) + \Gamma(s) \alpha(s))^\top dW_s. \end{aligned}$$

By the definition of δ_i^e and the fact that $\mathbb{1}_{A_1} + \mathbb{1}_{A_2} = 1$ on $[\tau, \theta]$, the first integral in the equation above is

$$\begin{aligned} &\int_\tau^{\cdot \wedge \theta} L(s) \left\{ \left(a(s) + \frac{\eta d(s)}{2(|d(s)| + 1)} \right) \mathbb{1}_{A_0 \cap A_1}(s) + \mathbb{1}_{A_0 \cap A_2}(s) \right. \\ &\quad \times \left. \left(a(s) + \frac{\eta d(s)}{2(|d(s)| + 1)} + \frac{2a(s)^+ + \eta}{\eta} \int_E \sum c_i^e(s) K_i(s, e) m_i(de) \right) \right\} ds. \end{aligned}$$

By (55), $a(s) \leq -\eta$ on $A_0 \cap A_1$. Then,

$$\left(a(s) + \frac{\eta d(s)}{2(|d(s)| + 1)} \right) \mathbb{1}_{A_0 \cap A_1}(s) \leq \left(-\eta + \frac{\eta}{2} \right) \mathbb{1}_{A_0 \cap A_1}(s) \leq 0. \quad (60)$$

By the definition of $A_{3,i}$ and (59), it holds that

$$\begin{aligned} & \mathbb{1}_{A_0 \cap A_2}(s) \left(a(s) + \frac{\eta d(s)}{2(|d(s)| + 1)} + \frac{2a(s)^+ + \eta}{\eta} \int_E \sum c_i^e(s) K_i(s, e) m_i(de) \right) \\ & \leq \mathbb{1}_{A_0 \cap A_2}(s) \left(a(s) + \frac{\eta}{2} - \frac{2a(s)^+ + \eta}{\eta} \cdot \frac{\eta}{2} \right) = -\mathbb{1}_{A_0 \cap A_2}(s) a(s)^-. \end{aligned} \quad (61)$$

Therefore, (60) and (61) imply that ΓL is a local super-martingale on $[\tau, \theta]$. Note that

$$\Gamma(\theta) - \Gamma(\theta-) = \int_E \bar{J}^{\nu(\theta), e}(\theta, X(\theta-), Y(\theta-), \tilde{\varphi})^\top \lambda(\{\theta\}, de).$$

Since $\tilde{\varphi} \in C(\mathbb{D})$ and (54) holds, $\tilde{\varphi}$ is locally bounded and globally bounded from above. This, together with (58) and the admissibility condition (2), implies that $\Gamma(\theta) - \Gamma(\theta-) \geq -K$ almost surely for some $K > 0$ (K may depend on $(t_0, x_0), \varepsilon, \nu$ and $\tilde{\varphi}$). Since $\Gamma(s) = Y(s) - \tilde{\varphi}(s, X(s)) - \kappa \geq -(\varepsilon + \kappa)$ on $[\tau, \theta]$, ΓL is bounded from below by a sub-martingale $-(\varepsilon + \kappa + K)L$ on $[\tau, \theta]$. This further implies that ΓL is a super-martingale by Fatou's Lemma. Since $\Gamma(\tau)L(\tau) < 0$ on $B \cap E_1$, the super-martingale property implies that there exists $F \subset B \cap E_1$ such that $F \in \mathcal{F}_\tau^t$ and $\Gamma(\theta \wedge \rho)L(\theta \wedge \rho) < 0$ on F . The non-negativity of L then yields $\Gamma(\theta \wedge \rho) < 0$. Therefore,

$$\begin{aligned} & Y(\theta_1) < \tilde{\varphi}(\theta_1, X(\theta_1)) + \kappa \text{ on } F \cap \{\theta_1 \leq \theta_2, \theta < \rho\}, \quad Y(\theta_2) < \tilde{\varphi}(\theta_2, X(\theta_2)) + \kappa \text{ on } F \cap \{\theta_1 > \theta_2, \theta < \rho\} \text{ and} \\ & Y(\rho) - (\tilde{\varphi}(\rho, X(\rho)) + \kappa) < 0 \text{ on } F \cap \{\theta \geq \rho\}. \end{aligned} \quad (62)$$

Since $(\theta_1, X(\theta_1)) \notin B_{\varepsilon/2}(t_0, x_0)$, it follows from the first two inequalities in (56) that

$$Y(\theta_1) < \tilde{\varphi}(\theta_1, X(\theta_1)) + \kappa < w(\theta_1, X(\theta_1)) \text{ on } F \cap \{\theta_1 \leq \theta_2, \theta < \rho\}. \quad (63)$$

On the other hand, since $Y(\theta_2) < \tilde{\varphi}(\theta_2, X(\theta_2)) + \kappa$ on $F \cap \{\theta_1 > \theta_2, \theta < \rho\}$ and (57) holds, $Y(\theta_2) - \tilde{\varphi}(\theta_2, X(\theta_2)) \leq -\varepsilon$ on $F \cap \{\theta_1 > \theta_2, \theta < \rho\}$. Observing that $(\theta_2, X(\theta_2)) \in B_{\varepsilon/2}(t_0, x_0)$ on $\{\theta_1 > \theta_2\}$, we get from the last inequality of (56) that

$$Y(\theta_2) - w(\theta_2, X(\theta_2)) < \tilde{\varphi}(\theta_2, X(\theta_2)) - \varepsilon - w(\theta_2, X(\theta_2)) < 0 \text{ on } F \cap \{\theta_1 > \theta_2, \theta < \rho\}. \quad (64)$$

From (63) and (64), we get that $Y(\theta) < w(\theta, X(\theta))$ on $F \cap \{\theta < \rho\}$. Therefore, from the definition of \mathbb{U}^- ,

$$\mathbb{P}(G_0 | F \cap \{\theta < \rho\}) > 0 \text{ if } \mathbb{P}(F \cap \{\theta < \rho\}) > 0. \quad (65)$$

From (62), it holds that

$$\mathbb{P}(G | F \cap \{\theta \geq \rho\}) > 0 \text{ if } \mathbb{P}(F \cap \{\theta \geq \rho\}) > 0. \quad (66)$$

Since $G_0 \subset G$, (65) and (66) imply that $\mathbb{P}(G \cap F) > 0$. Therefore, $\mathbb{P}(G \cap B \cap E_1) > 0$. \square

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